Assignment 5

***Due**: Wednesday, Nov 30th, 2022 @ 5:00PM*

CPSC 447/547 Introduction to Quantum Computing (Fall 2022)

1 Introduction

Welcome to Assignment 5 for CPSC 447/547 (Introduction to Quantum Computing). As usual, collaboration is encouraged; if you discussed with anyone besides the course staff about the assignment, *please list their names* in your submission.

Getting Started.

This assignment has *only one* part, the *written portion*. Typesetting your solutions to the written portion is not mandatory but highly encouraged. See the instructor's note on Ed for details about Latex for quantum computing. To start,

- Create a folder for Assignment 5, e.g., `A5/`
- Download the starter files for this assignment to that folder from the course website:
  - `A5.pdf`
  - `written.tex`
- Write your solutions in `written.tex`

New Structure.

There are now two types of tasks, *mandatory* or *optional*. *Only the mandatory tasks will be counted towards your final grade for this assignment.* Different from the mandatory tasks, the optional tasks will be marked with "**($\star$ pts)**".

Submission.

Once you have completed and are ready to submit, upload to Gradescope (accessed through Canvas): `written.pdf`. Late submissions (for up to two days) will receive a 50% penalty. Your written solution will be graded manually by our course staff.

WRITTEN PORTION

This portion of the assignment has a total of 100 *points.*

In this assignment, we will explore the basics of quantum error correction. In the first part of the assignment, we will use the operator sum representation to understand some important single-qubit quantum errors. Then in the second part, we introduce the stabilizer language to describe quantum error correction codes.

2 Phase Flip Channel

Task 2.1 (24 pts)

Consider a single-qubit error model where the qubit experiences a phase flip, Z, with probability p, and stays the same otherwise. Specifically, if the input quantum state is ρ_{in}, then the output quantum state is:

$$\rho_{out} = pZ\rho_{in}Z^{\dagger} + (1-p)\rho_{in} \tag{1}$$

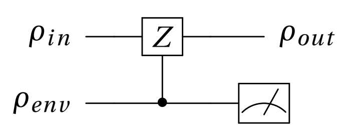
(a) Suppose $\rho_{in} = |\psi\rangle\langle\psi|$ where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. What is ρ_{out}? Write your answer as a 2×2 density matrix, in terms of parameter p.

(b) The effect of an error on a quantum system can be mathematically represented by a quantum channel, $\mathscr{E}(\rho)$, defined as

$$\mathscr{E}(\rho) = \sum_k E_k \rho E_k^{\dagger}$$

where the E_k are called the Kraus operators satisfying the condition $\sum_k E_k^{\dagger} E_k = I$. Give the set of two Kraus operators $\{E_0, E_1\}$ describing the phase flip channel.

(c) Recall from lecture that one can model such error as an interaction of the quantum system with an environment. The Kraus operators will then arise from tracing out the environment. To see this, we construct a quantum circuit model for the phase flip channel as follows:



Here, the top qubit is the input quantum system, and the bottom qubit is the environment. Give the environment state ρ_{env} such that the circuit is equivalent to the phase flip channel on the top qubit ρ_{in}. Write ρ_{env} as a density matrix in terms of the parameter p. (Hint: we want to initialize ρ_{env} such that a Z gate is performed on ρ_{in} with probability p.)

3 Dephasing Channel

Task 3.1 (24 pts)

Dephasing channel is an important physical process in quantum systems; it does not make any transitions in the $\{|0\rangle, |1\rangle\}$ basis, but instead changes the relative phase between $|0\rangle$ and $|1\rangle$.

In the operator sum representation, we consider for a single-qubit input state ρ, a dephasing channel is defined as

$$\mathscr{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

where $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}$.

(a) Consider the input quantum state $\rho_{in} = |+\rangle\langle+|$. Give the output quantum state $\rho_{out} = \mathscr{E}(\rho_{in})$. Write your answer as a 2×2 matrix in terms of λ.

(b) It turns out we can write the two Kraus operators as a linear combination of the Pauli operators, I and Z. In particular, find the coefficients, a, b, c, d, such that $E_0 = aI + bZ$ and $E_1 = cI + dZ$. Write your answers in terms of λ.

(c) Use your solution to part (b) to rewrite the dephasing channel. Show that you can write $\mathscr{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = pZ\rho Z^\dagger + (1-p)\rho$ for some p. Write p in terms of λ.

(d) (**Optional**) In real quantum systems, dephasing occurs continuously in time. Imagine r is the rate of dephasing, so the dephasing parameter $\lambda = r\Delta T \ll 1$ for some small interval of time ΔT. At time $T = n\Delta T$, we can model the resulting quantum state as applying the dephasing channel n times: $\mathscr{E}^n(\rho)$. Consider the input quantum state $\rho_{in} = |+\rangle\langle+|$. To model a continuous dephasing process up to some constant time T, we divide T into $n \to \infty$ segments, each applied with $\mathscr{E}$ for infinitesimal duration ΔT. Give the output quantum state at some fixed time T: $\rho_{out}(T) = \mathscr{E}^n(\rho_{in})$, for $n \to \infty$. Write your answer as a 2×2 matrix for ρ_{out} in terms of r and T.

4 Erasure and Depolarizing Channel

Task 4.1 (⋆ pts)

Recall from lecture, we call a quantum state $\rho = \frac{I}{2}$ the *maximally mixed state* because it describes a quantum state at the origin of the Bloch sphere, which can be understood as a completely classical random state: $|0\rangle$ and $|1\rangle$ with equal probability.

Suppose a quantum channel maps *any* input quantum state to the maximally mixed state. This defines the erasure channel:

$$\mathscr{E}(\rho_{in}) = \frac{I}{2},$$

which completely destroys the input qubit ρ_{in}.

(a) Consider a generic quantum state $\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Give the density matrices for $X\rho X, Y\rho Y, Z\rho Z$. Write your answers in terms of a, b, c, d.

(b) Show that $I = \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z)$ for any single-qubit density operator ρ.

(c) Give the set of Kraus operators $\{E_0, E_1, E_2, E_3\}$ for the erasure channel.

(d) The depolarizing channel describes a random erasure process. Under this channel, a input quantum state is erased to the maximally mixed state $\frac{I}{2}$ with probability p and stays the same otherwise:

$$\mathscr{E}(\rho) = p\frac{I}{2} + (1-p)\rho$$

Give the set of Kraus operators $\{E_0, E_1, E_2, E_3\}$ that describes the depolarizing channel.

5 7-Bit Hamming Code

Task 5.1 ($\star$ pts)

Recall from lecture, we define a classical linear code using a $k \times n$ generator matrix G and a $(n-k) \times n$ parity check matrix H to encode k bits of information using n bits. The codewords are defined as the column vectors $\vec{x}_v = G^T \vec{v}$, where $\vec{v}$ is the column vector representing a k-bit integer, ranging from 0 to $2^k - 1$. Note that all arithmetic operations are done with modulo 2. Errors can be detected by performing parity checks specified by the rows of a $(n-k) \times n$ matrix H. A good parity check matrix satisfies the condition that $H\vec{x} = 0$ for all codewords $\vec{x}$.

For example, the Hamming code encodes $k = 4$ bits using $n = 7$ bits. The generator matrix G is defined as

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The parity check matrix H is defined as

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

(a) Give the column vector for the codeword $\vec{x}_0 = G^T\vec{0}$. Here $\vec{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$.

(b) Give the column vector for the codeword $\vec{x}_3 = G^T\vec{3}$. Here $\vec{3} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$.

(c) Give the column vector for the codeword $\vec{x}_{15} = G^T\vec{15}$. Here $\vec{15} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

(d) Verify that $H\vec{x}_{15} = 0$.

(e) Consider an error $\vec{e}$ which occurs to one of the codewords, resulting in

$$\vec{y} = \vec{x} + \vec{e} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

What are the results of the parity checks: $H\vec{y}$?

(f) Consider an error $\vec{e}$ which occurs to one of the codewords, resulting in $\vec{y} = \vec{x} + \vec{e}$. Suppose the results of the parity checks: $H\vec{y} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. If we assume that $\vec{e}$ is a single bit-flip, can we locate the bit flip? If so, give $\vec{e}$. If not, explain why.

6 7-Qubit Steane Code

Task 6.1 (52 pts)

The 7-qubit code (also known as Steane code) is constructed from the classical 7-bit Hamming code C. The Steane code encodes 1 qubit of information using 7 qubits. In this question, we will explore this construction and its properties.

(a) For each codeword x of C, we define a 7-qubit quantum state $|x\rangle$. For example, if $x = \begin{bmatrix}1 & 1 & 0 & 0 & 1 & 1 & 0\end{bmatrix}^T$, then $|x\rangle = |1100110\rangle$. For each row h of the parity check matrix H, we define a Z-type stabilizer operator S_h. For example, if $H_h = \begin{bmatrix}1 & 1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$, then $S_h = Z \otimes Z \otimes Z \otimes Z \otimes I \otimes I \otimes I$. Show that $|1100110\rangle$ is a $(+1)$-eigenstate of all three stabilizers. That is, $S_h|1100110\rangle = |1100110\rangle$ for each S_h.

(b) A single-qubit bit-flip error can be diagnosed by the above stabilizers. Consider a non-codeword $\left|y\right\rangle = |0110010\rangle$. Show that $\left|y\right\rangle$ is a (-1)-eigenstate of at least one of the stabilizers.

(c) In order to also diagnose phase flip errors, we group the codewords into two subsets: $E = \{|x\rangle : x \in C \text{ and } x \text{ has even number of } 1\}$ and $O = \{|x\rangle : x \in C \text{ and } x \text{ has odd number of } 1\}$. List the elements in E and O respectively.

(d) We define the two codewords of the 7-qubit Steane code as follows:

$$|0\rangle_L = \frac{1}{\sqrt{8}} \sum_{x \in E} |x\rangle, \qquad |1\rangle_L = \frac{1}{\sqrt{8}} \sum_{x \in O} |x\rangle$$

We also define an additional set of stabilizers: For each row h of the parity check matrix H, we define an X-type stabilizer operator S'_h. For example, if $H_h = \begin{bmatrix}1 & 1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$, then $S'_h = X \otimes X \otimes X \otimes X \otimes I \otimes I \otimes I$. Show that $S'_h|0\rangle_L = |0\rangle_L$ and $S'_h|1\rangle_L = |1\rangle_L$ for all three S'_h.

(e) Consider an initial quantum state $\left|\psi\right\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L$. Due to some single-qubit bit-flip or phase flip error, the quantum state results in $\left|\psi'\right\rangle$. Suppose the stabilizer syndrome measurements have the following outcome:

$$S_0\left|\psi'\right\rangle = \left|\psi'\right\rangle, S_1\left|\psi'\right\rangle = \left|\psi'\right\rangle, S_2\left|\psi'\right\rangle = \left|\psi'\right\rangle, S'_0\left|\psi'\right\rangle = \left|\psi'\right\rangle, S'_1\left|\psi'\right\rangle = -\left|\psi'\right\rangle, S'_2\left|\psi'\right\rangle = \left|\psi'\right\rangle.$$

What was the error?