Assignment 4

Due: *Wednesday, Nov 15th, 2023 @ 5:00PM*

CPSC 447/547 Introduction to Quantum Computing (Fall 2023)

1 Introduction

Welcome to Assignment 4 for CPSC 447/547 (Introduction to Quantum Computing). As usual, collaboration is encouraged; if you discussed with anyone besides the course staff about the assignment, *please list their names* in your submission.

Getting Started.

This assignment has only one part, the written portion. Typesetting your solutions to the written portion is not mandatory but highly encouraged. See the instructor's note on Ed for details about Latex for quantum computing. To start,

- Create a folder for Assignment 4, e.g., `A4/`
- Download the starter files for this assignment to that folder from the course website:
  - `A4.pdf`
  - `written.tex`
- Write your solutions in `written.tex`

New Structure.

There are now two types of tasks, mandatory or optional. *Only the mandatory tasks will be counted towards your final grade for this assignment.* Different from the mandatory tasks, the optional tasks will be marked with "**($\star$ pts)**".

Submission.

Once you have completed and are ready to submit, upload to Gradescope (accessed through Canvas): `written.pdf`. Late submissions (for up to two days) will receive a 50% penalty. Your written solution will be graded manually by our course staff.

WRITTEN PORTION

This portion of the assignment has a total of 100 *points.*

2 Grover's Algorithm

Task 2.1 (15 pts)

Consider a function $f : \{0,1\}^4 \to \{0,1\}$. In the standard Grover's algorithm, we assume we know the number of inputs such that $f(x) = 1$. Suppose there are $K = 2$ of those satisfied inputs: $x = 8 = 1000$ and $x = 9 = 1001$.

(a) Given an oracle to the above function, $O_f^{\pm}$, we want to find those inputs x such that $f(x) = 1$. What is the optimal number of iterations in Grover's algorithm? That is, after how many iterations of oracle access is the probability of measuring $|1000\rangle$ or $|1001\rangle$ maximized? (Write the exact number, not in the big-O notation.)

(b) After the optimal number of iterations, what is the probability that $|1001\rangle$ is measured? (Write the exact number, not in big-O.)

Task 2.2 (10 pts)

Following the previous question,

(a) Suppose we apply one more iteration than the optimal number, what is the probability that $|1001\rangle$ is measured? (Write the exact number, not in big-O.)

3 Density Operators

Task 3.1 ($\star$ pts)

Recall from lecture, we define the density operator for an n-qubit quantum state as a Hermitian matrix $\rho \in \mathbb{C}^{2^n \times 2^n}$ where $\rho \succcurlyeq 0$ (positive semi-definite) and $\mathrm{Tr}(\rho) = 1$ (unit trace).

(a) Show that if $\langle v|\rho|v\rangle \geq 0$ holds for all unit vectors $|v\rangle$, then $\rho \succcurlyeq 0$.

Task 3.2 (30 pts)

A density operator can be used to represent an ensemble of quantum states. Suppose the probability of sampling $|\psi_j\rangle$ is p_j, then the matrix for the density operator is

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

(a) Suppose we have a pure state, e.g., $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ with probability 1. What is its density matrix ρ_a? (Write down the matrix explicitly.)

(b) Suppose we have a mixed state, e.g., $|0\rangle$ with probability $\frac{1}{2}$ and $|1\rangle$ with probability $\frac{1}{2}$. What is its density matrix ρ_b? (Write down the matrix explicitly.)

(c) Suppose we have another mixed state, e.g., $|0\rangle$ with probability $\frac{1}{2}$ and $|+\rangle$ with probability $\frac{1}{2}$. What is its density matrix ρ_c? (Write down the matrix explicitly.)

(d) Recall from lecture, in the Bloch sphere picture, we can locate a density matrix $\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$ with Cartesian coordinate (r_x, r_y, r_z). Write down the coordinates for ρ_a, ρ_b, ρ_c from previous questions.

(e) Notice that pure states are on the surface of the Bloch sphere and mixed states are at the interior of the Bloch sphere. We can define "how pure" a quantum state is by how close ρ is to the surface. Define $r = \sqrt{r_x^2 + r_y^2 + r_z^2}$ to be the distance from the origin to ρ. Show that $\mathrm{Tr}(\rho^2) = \frac{1}{2}(1 + r^2)$.

(f) We can use $\mathrm{Tr}(\rho^2)$ to evaluate the "purity" of n-qubit quantum states. Here $\rho \in \mathbb{C}^{2^n \times 2^n}$. What is the lower bound and the upper bound of $\mathrm{Tr}(\rho^2)$? Also write down two example quantum states, $\rho_{lower}, \rho_{upper}$, that saturate the bounds respectively.

4 Depolarizing Channel

Task 4.1 (20 pts)

Recall from lecture, we call a quantum state $\rho = \frac{I}{2}$ the *maximally mixed state* because it describes a quantum state at the origin of the Bloch sphere, which can be understood as a completely classical random state: $|0\rangle$ and $|1\rangle$ with equal probability.

Suppose a quantum channel maps *any* input quantum state to the maximally mixed state. This defines the erasure channel:

$$\mathscr{E}(\rho_{in}) = \frac{I}{2},$$

which completely destroys the input qubit ρ_{in}.

(a) Consider a generic quantum state $\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Give the density matrices for $X\rho X, Y\rho Y, Z\rho Z$. Write your answers in terms of a, b, c, d.

(b) Show that $I = \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z)$ for any single-qubit density operator ρ.

(c) Give the set of Kraus operators $\{E_0, E_1, E_2, E_3\}$ for the erasure channel.

(d) The depolarizing channel describes a random erasure process. Under this channel, a input quantum state is erased to the maximally mixed state $\frac{I}{2}$ with probability p and stays the same otherwise:

$$\mathscr{E}(\rho) = p\frac{I}{2} + (1-p)\rho$$

Give the set of Kraus operators $\{E_0, E_1, E_2, E_3\}$ that describes the depolarizing channel.

5 Entropy

Task 5.1 ($\star$ pts)

In information theory, entropy is a quantity commonly used to measure the level of "uncertainty" in a random variable's outcomes. For example, if we have a discrete random variable

x, such that x has outcome $x_1, x_2, \ldots, x_n$ with probability $p_1, p_2, \ldots, p_n$ respectively, then we can define the entropy of x as

$$H(x) = -\sum_{j=1}^{n} p_j \log(p_j).$$

Here the log function has base 2, so the entropy is in the unit of bits [1]. In the special case where a random variable has two outcomes, such as a coin flip where the probability of heads is p and the probability of tails is $1-p$. We define the binary entropy as

$$H(p) = -p\log(p) - (1-p)\log(1-p).$$

(a) Suppose Alice flips a fair coin, i.e., getting the outcome of heads ($x = H$) or tails ($x = T$) with equal probability. What is $H(x)$?

(b) Consider Alice and Bob play a series of 5 games. First player who wins 3 games wins the series. For example, the series might end with outcome AAA, or BAAA, or BBAAB. Assuming either player has equal probability of winning a game (with no ties). Let x be the random variable for the number of games played in a series. What is $H(x)$?

Task 5.2 (25 pts)

Recall from lecture, our definition of density matrix ρ is analogous to that of probability density $p = \{p_1, p_2, p_3, \ldots, p_n\}$. For example, the positive condition in $\rho \succcurlyeq 0$ is analogous to $p \geq 0$ and the unity condition $\mathrm{Tr}(\rho) = 1$ is analogous to $\sum_j p_j = 1$.

We can therefore define a similar measure of uncertainty in qubits. This is called the von Neumann entropy:

$$S(\rho) = -\mathrm{Tr}(\rho\log(\rho)),$$

where $\log(\rho)$ is defined by Spectral theorem and applying the log function to the eigenvalues: $\log(\rho) = \sum_j \log(\lambda_j)\,|\psi_j\rangle\langle\psi_j|$.

(a) For a single qubit pure state. e.g., $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ with probability 1. What is its entropy $S(\rho_a)$?

(b) For a mixed state, e.g., $|0\rangle$ with probability $\frac{1}{2}$ and $|1\rangle$ with probability $\frac{1}{2}$. What is its entropy $S(\rho_b)$?

(c) For another mixed state, e.g., $|0\rangle$ with probability $\frac{1}{2}$ and $|+\rangle$ with probability $\frac{1}{2}$. What is its entropy $S(\rho_c)$?

(d) For an arbitrary single-qubit mixed state with density matrix $\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$, that is with Cartesian coordinate (r_x, r_y, r_z) in Bloch sphere. What are the eigenvalues of ρ? (Write your answer in terms of r_x, r_y, r_z.)

(e) Following the previous question, what is $S(\rho)$? (Write your answer in terms of r_x, r_y, r_z.)

[1] https://en.wikipedia.org/wiki/Entropy_(information_theory)