Yale University

Quantum Gates

PART A

CPSC 4470/5470

Introduction to Quantum Computing

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Mathematical Model of Quantum Computing

Four Principles to model quantum systems mathematically:

1. **Superposition:**
   The state of a qubit is a normalized complex vector in the two-dimensional Hilbert Space.
2. **Composition:**
   The joint state of many (independent) quantum systems is the tensor product of component states.
3. **Transformation:**
   Time evolution of a quantum system is a unitary process.
4. **Measurement:**
   Readout information from a quantum state causes the superposition state to collapse/project to one of its basis states randomly.

Niels Bohr: *"Anyone who is not shocked by quantum theory has not understood it."*

A Qubit – The Bloch Sphere

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$

- Normalized: $|\alpha|^2 + |\beta|^2 = 1$
- Global phase does not matter:
  $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ not distinguishable

Derive on board (in terms of two real numbers):

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix} a \\ e^{i\phi}\sqrt{1-a^2}\end{bmatrix} \quad 0 \le a \le 1, 0 \le \phi < 2\pi$$

We can visualize two real numbers!

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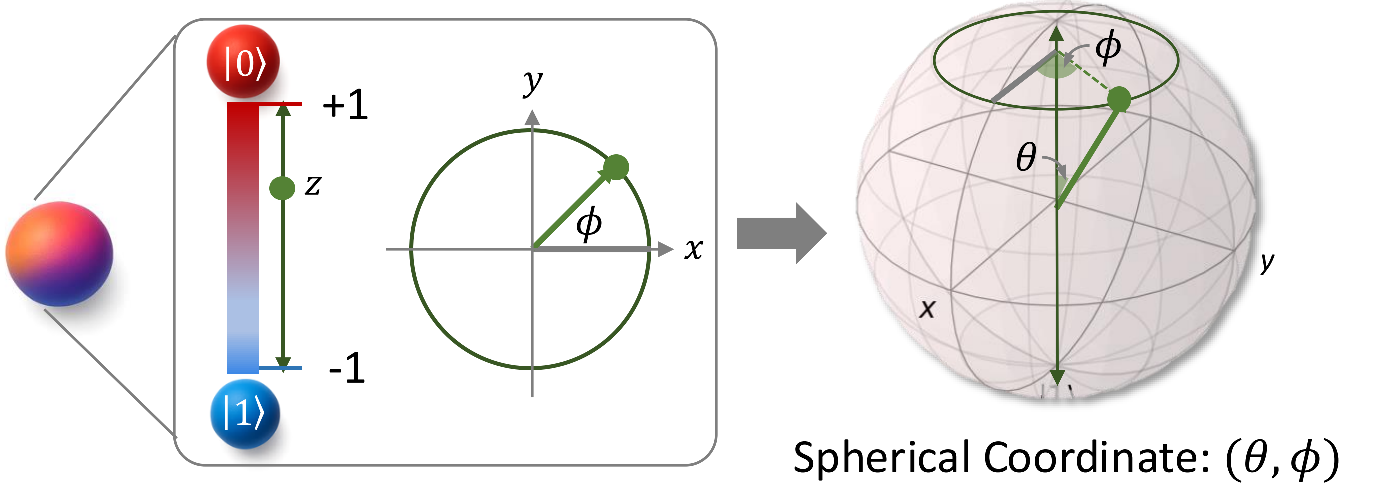
$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix} a \\ e^{i\phi}\sqrt{1-a^2}\end{bmatrix}$$

$$(0 \le a \le 1, 0 \le \phi < 2\pi)$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\end{bmatrix} \quad (0 \le \theta \le \pi, 0 \le \phi < 2\pi)$$

We can visualize two real numbers:

- $a^2 \in [0,1]$: Probability of measuring $|0\rangle$
  - $z = 2\left(a^2 - \frac{1}{2}\right) \in [-1,1]$ $= 2\left(\cos^2\frac{\theta}{2} - \frac{1}{2}\right) = \cos\theta$
- $e^{i\phi}$: relative phase, $0 \le \phi < 2\pi$



Spherical Coordinate: (θ, ϕ)

A Qubit – The Bloch Sphere

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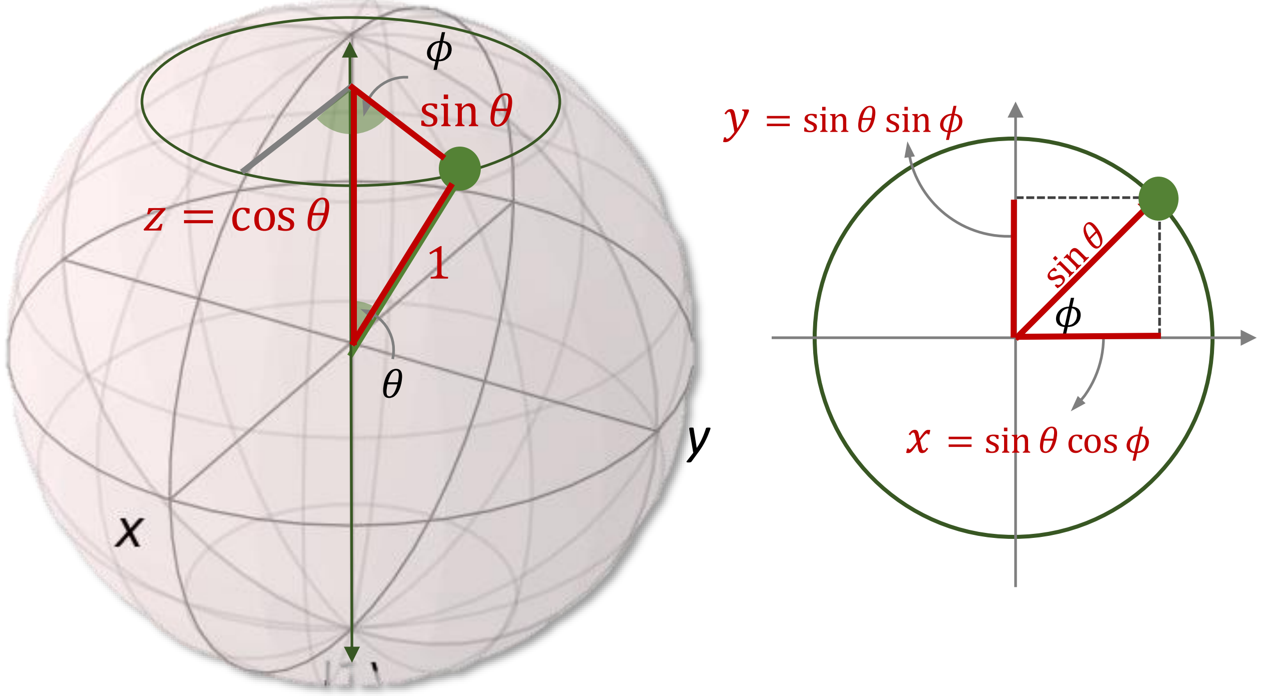
$$= \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$$



Spherical Coordinate: (θ, ϕ)

What about its **Cartesian coordinate**: (x, y, z)?

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Where are $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$ on Bloch sphere?

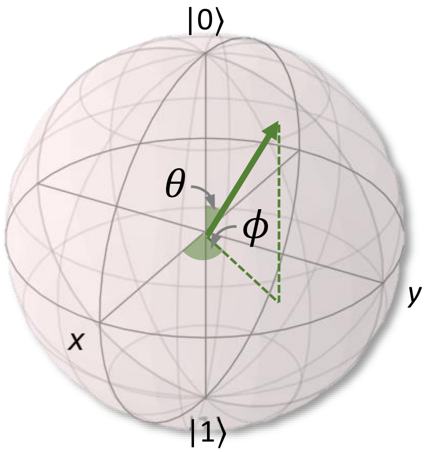
Density operator: $|\psi\rangle\langle\psi| = \frac{1}{2}(\sigma_I + x\cdot\sigma_X + y\cdot\sigma_Y + z\cdot\sigma_Z)$

Single-Qubit Pauli Gates

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$

$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \sim \begin{bmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{bmatrix}$: two real numbers

A point on the surface of the **Bloch sphere**:



Pauli gates:

- $\sigma_I = I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
- $\sigma_X = X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
- $\sigma_Y = Y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$
- $\sigma_Z = Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

Transformations:

$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \xrightarrow{\mathrm{I}} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

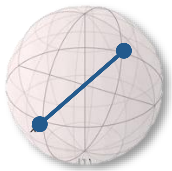
$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \xrightarrow{X} \begin{bmatrix}\beta\\ \alpha\end{bmatrix}$

$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \xrightarrow{Y} \begin{bmatrix}-i\beta\\ \alpha\end{bmatrix}$

$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \xrightarrow{Z} \begin{bmatrix}\alpha\\ -\beta\end{bmatrix}$

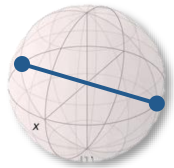
Eigenstates:

$U|\psi\rangle = \lambda|\psi\rangle$



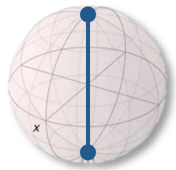
$|+\rangle$ and $|-\rangle$

(x axis)



$|+i\rangle$ and $|-i\rangle$

(y axis)



$|0\rangle$ and $|1\rangle$

(z axis)

Geometric Interpretations of Pauli Gates

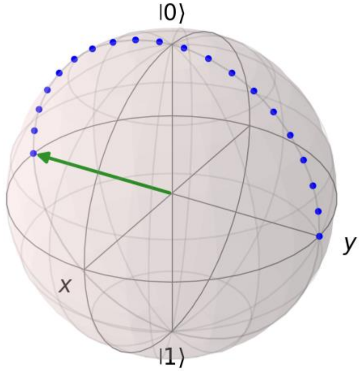
Pauli Gate: $\sigma_X = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_Y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_Z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Transformation: $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \xrightarrow{X} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \xrightarrow{Y} \begin{bmatrix} -i\beta \\ \alpha \end{bmatrix}$ $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \xrightarrow{Z} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$

Change in (θ, ϕ)?
(Derive on board)

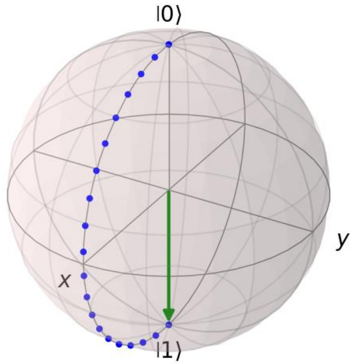
$(\theta, \phi) \xrightarrow{X} (\pi - \theta, -\phi)$ $(\theta, \phi) \xrightarrow{Z} (\theta, \pi + \phi)$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \sim \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$



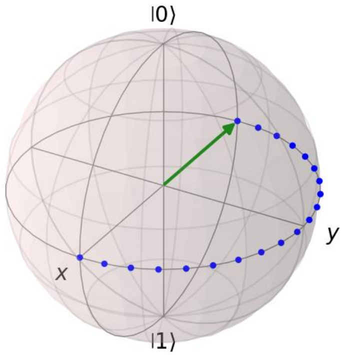
$R_X(\pi)$

Rotation about x-axis by 180°



$R_Y(\pi)$

Rotation about y-axis by 180°



$R_Z(\pi)$

Rotation about z-axis by 180°

Closer look: Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors in QM

Symmetric matrix (real):

$$S^T = S$$

Hermitian matrix (complex):

$$H^\dagger = H$$

- Hermitian matrix has **real eigenvalues.**
- Corresponding to *physical observable* with real-valued quantity.

Why?

Eigenvalue equation for a linear operator A:

$$A|v_j\rangle = \lambda_j|v_j\rangle$$

where $|v_j\rangle$ is the (non-zero) **eigenvector**,
and λ_j is a complex number known as the **eigenvalue**.

<u>Derive on board</u> (Pauli Matrices):

"Pauli Z operator" $\sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- Eigenvalues: $\lambda_0 = 1$ and $\lambda_1 = -1$
- Eigenvectors: $|v_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|v_1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Standard basis! $\{|0\rangle, |1\rangle\}$

"Pauli Y operator" $\sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

- Eigenvalues: $\lambda_0 = 1$ and $\lambda_1 = -1$
- Eigenvectors: $|v_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\{|+i\rangle, |-i\rangle\}$ basis!

"Pauli X operator" $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- Eigenvalues: $\lambda_0 = 1$ and $\lambda_1 = -1$
- Eigenvectors: $|v_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\{|+\rangle, |-\rangle\}$ basis!

Spectral Theorem

For a linear operator that is normal ($A^\dagger A = AA^\dagger$), we can write it in the **spectral decomposition**:

$$A = \sum_j \lambda_j |v_j\rangle\langle v_j|$$

where λ_j are the eigenvalues, and $|v_j\rangle$ are the corresponding (orthonormal) eigenvectors.

Examples:

$$\sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1| \qquad \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (+1)|+\rangle\langle +| + (-1)|-\rangle\langle -|$$

Applications:

- Power of a matrix:

$$A^8 = \left(\sum_j \lambda_j |v_j\rangle\langle v_j|\right)^8 = \sum_j \lambda_j^8 |v_j\rangle\langle v_j|$$

- Exponential of a matrix:

$$e^A \equiv \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_j e^{\lambda_j} |v_j\rangle\langle v_j|$$

Example:

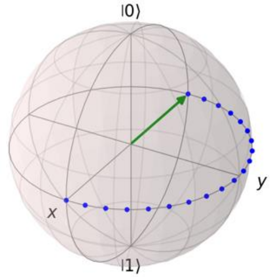
$$e^{i\theta\sigma_z} = e^{i\theta}|0\rangle\langle 0| + e^{-i\theta}|1\rangle\langle 1| = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

$= \cos\theta\, \sigma_I - i \sin\theta\, \sigma_Z$ (by Euler's formula)

Pauli Rotation Gates

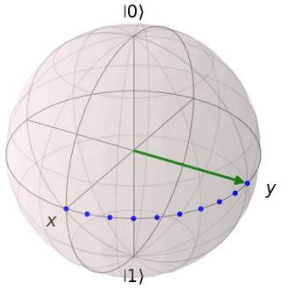
Pauli-Z gate:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$R_Z(\pi)$

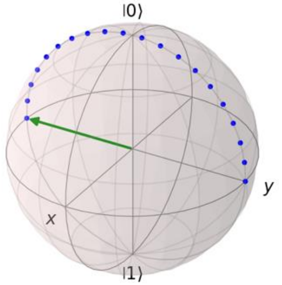
What about $R_Z\left(\frac{\pi}{2}\right)$?



$$R_Z\left(\frac{\pi}{2}\right) = \sqrt{Z} = S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

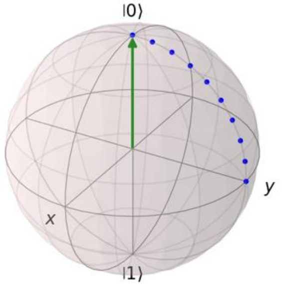
Pauli-X gate:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$R_X(\pi)$

What about $R_X\left(\frac{\pi}{2}\right)$?



Spectral theorem!

$$R_X\left(\frac{\pi}{2}\right) = \sqrt{X} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} e^{\frac{i\pi}{4}}$$

Global phase.

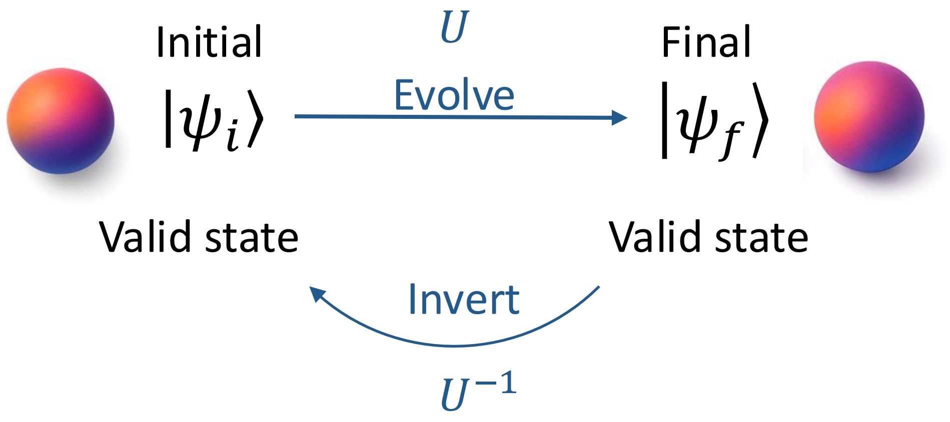
Rotation gates:

$$\begin{cases} R_X(\theta) = e^{-\frac{i\theta}{2}X} = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) X \\ R_Y(\theta) = e^{-\frac{i\theta}{2}Y} = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) Y \\ R_Z(\theta) = e^{-\frac{i\theta}{2}Z} = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) Z \end{cases}$$

(Using spectral theorem and Euler's formula)

Principle #3 – Transformation

Unitary Transformation: The evolution of a quantum state can be described as a *norm-preserving linear transformation* (a.k.a. unitary matrix).



- **Norm-preserving**:

$$\| |\psi_i\rangle \|^2 = \left\| |\psi_f\rangle \right\|^2 = 1$$

- **Linear transformation**:

Linear operator: $|\psi_f\rangle = U|\psi_i\rangle$, for some matrix U.

"Preserves inner product."

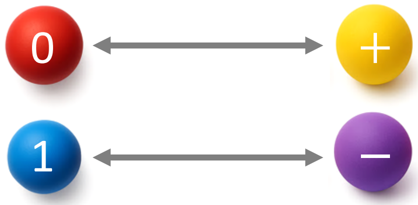
Why unitary? $U^\dagger U = I$

- The process is **reversible** and **deterministic**: $U^{-1} = U^\dagger$
- In physics: "Coherent process"

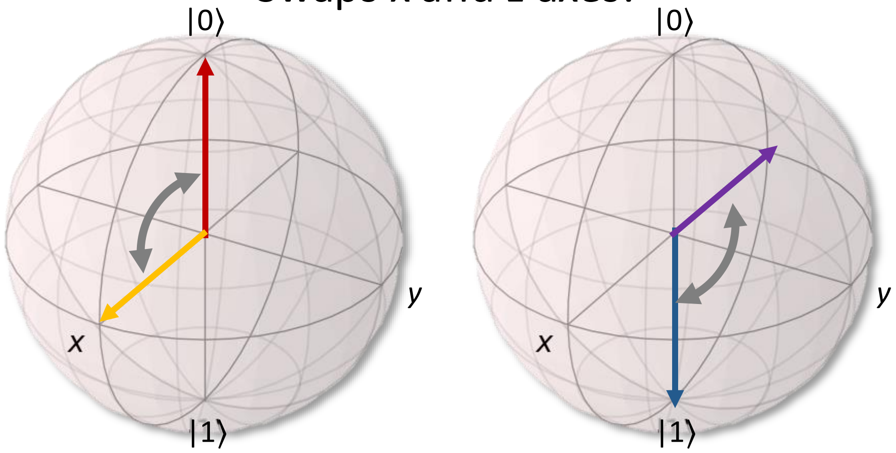
What about Hadamard Gate?

Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

- $H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$
- $H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle$
- $H|+\rangle = |0\rangle$
- $H|-\rangle = |1\rangle$

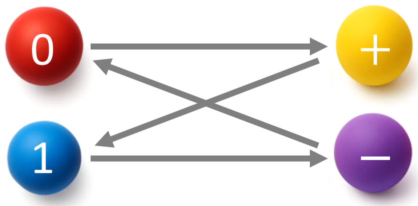


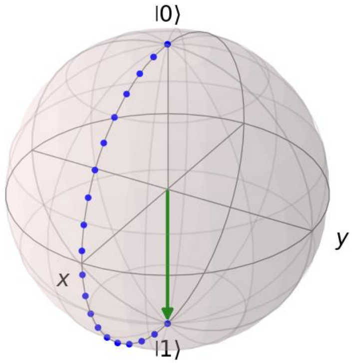
“Swaps x and z axes!”



Compared to **y-axis rotation**: $R_Y\left(\frac{\pi}{2}\right)$

- $R_Y\left(\frac{\pi}{2}\right)|0\rangle = |+\rangle$
- $R_Y\left(\frac{\pi}{2}\right)|1\rangle = |-\rangle$
- $R_Y\left(\frac{\pi}{2}\right)|+\rangle = |1\rangle$
- $R_Y\left(\frac{\pi}{2}\right)|-\rangle = |0\rangle$





$R_Y(\pi)$

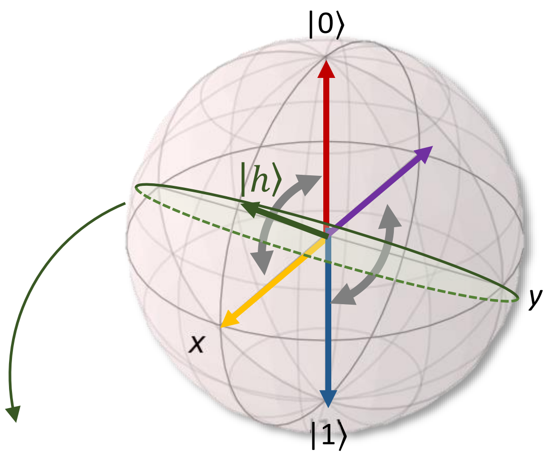
Understanding Hadamard

Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Take another input state:

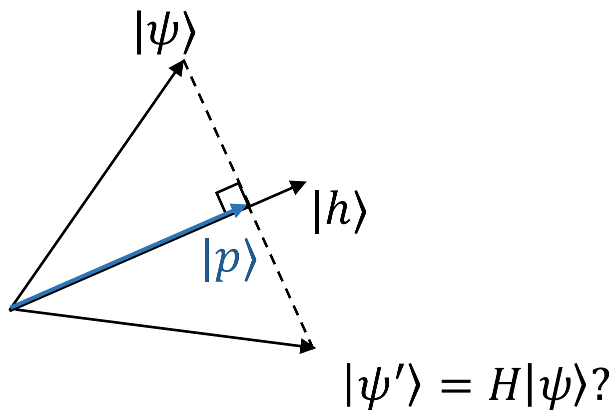
$$|h\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle$$

Check: $H|h\rangle = ?$



Reflection about the plane at 45 between x and z axes.

Why is it a **reflection** about $|h\rangle$?



Derive on board: **What is the reflected state?**

We have $|p\rangle = \Pi_h|\psi\rangle, \Pi_h = |h\rangle\langle h|$

$$|\psi'\rangle = (I - 2\Pi_h)|\psi\rangle$$

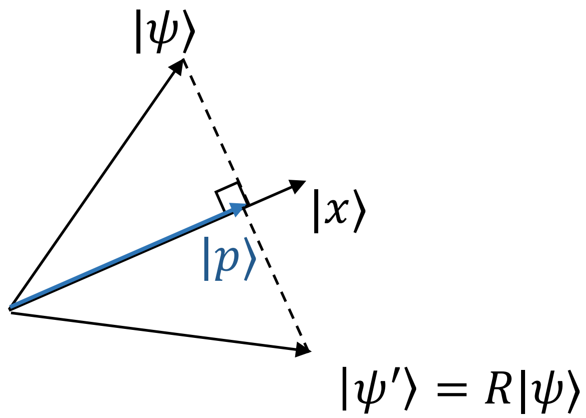
Indeed, $(I - 2\Pi_h) = H$!

Reflections

More generally, given a projector $\Pi = |\mathrm{x}\rangle\langle x|$, the projected state $|p\rangle = \Pi|\psi\rangle = p|x\rangle$

We can define a **reflection operator:**

$$R = I - 2\Pi$$



<u>Derive on board</u>:

- $R^2 = \ ?$
- For a vector $|v\rangle$ in the projected "plane" ($\Pi|\mathrm{v}\rangle = |v\rangle$): what is $R|v\rangle$?
- For a vector $|v^{\perp}\rangle$ orthogonal to the projected "plane" ($\Pi|v^{\perp}\rangle = 0$): what is $R|v^{\perp}\rangle$?