Yale University

Quantum Measurements

CPSC 4470/5470

Introduction to Quantum Computing

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Mathematical Model of Quantum Computing

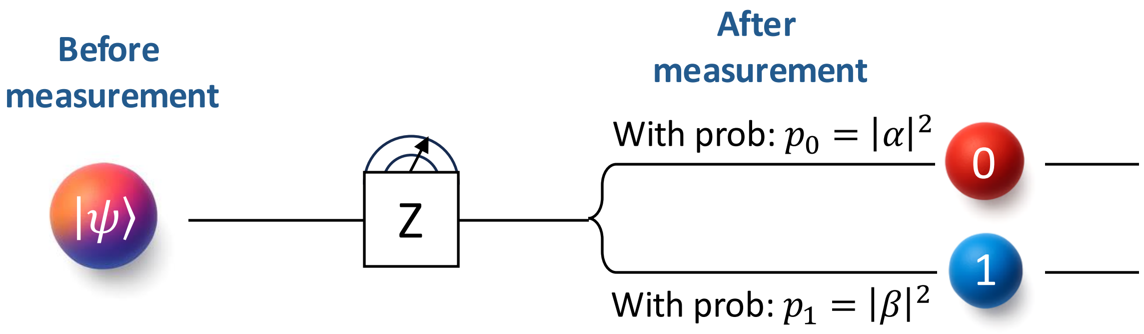
Four Principles to model quantum systems mathematically:

1. **Superposition:**
   The state of a qubit is a normalized complex vector in the two-dimensional Hilbert Space.
2. **Composition:**
   The joint state of many (independent) quantum systems is the tensor product of component states.
3. **Transformation:**
   Time evolution of a quantum system is a unitary process.
4. **Measurement:**
   Readout information from a quantum state causes the superposition state to collapse/project to one of its basis states randomly.

P Jordan: *"Observations not only disturb what has to be measured, they produce it…*
We compel [the electrons] to assume their definite position."

Principle #4: Measuring Qubits

The state of a quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is *not* directly observable.
Measuring $|\psi\rangle$ (in the Z basis) collapses it to either $|0\rangle$ or $|1\rangle$, with probability $|\alpha|^2$ and $|\beta|^2$, respectively.

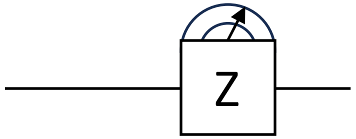


The measurement process is probabilistic and irreversible and generally disturbs the quantum system.

Physically: E.g., observing an atom in its ground energy level ($|0\rangle$) or its excited energy level ($|1\rangle$).

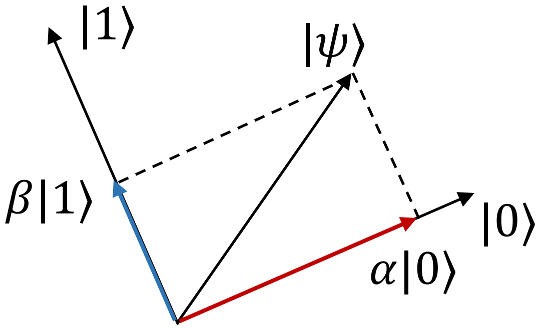
Understanding Measurements as Projections

Measurement along the **z basis** (“standard basis” or “computational basis”):



Each basis has a corresponding **projector**:

$$\Pi_0 = |0\rangle\langle 0|, \qquad \Pi_1 = |1\rangle\langle 1|$$



$|\psi\rangle$ **projects** to either $|0\rangle$ or $|1\rangle$

- Probability of measuring $|0\rangle$:

  $$|\langle 0|\psi\rangle|^2$$

- Probability of measuring $|1\rangle$:

  $$|\langle 1|\psi\rangle|^2$$

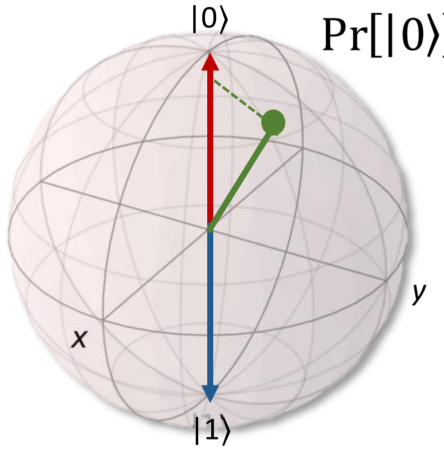
<u>Derive on board</u>:

- What are the projected states?
  - $\Pi_0|\psi\rangle$
  - $\Pi_1|\psi\rangle$

  Collapsed states (how to normalize?)

- What about the inner products?
  - $\langle\psi|\Pi_0|\psi\rangle$
  - $\langle\psi|\Pi_1|\psi\rangle$

  Probabilities



$$\Pr[|0\rangle] = \frac{z+1}{2}$$

Observable as a Hermitian Operator

For the standard-basis measurement, we take the Pauli Z operator:

$$Z = (+1)\Pi_0 + (-1)\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ (Spectral theorem)}$$

If we define a **random variable** z that takes Z's eigenvalues:

$$z = \begin{cases} +1 \text{ if } |\psi\rangle \text{ collapsed to } |0\rangle \text{ (with prob. } \langle\psi|\Pi_0|\psi\rangle) \\ -1 \text{ if } |\psi\rangle \text{ collapsed to } |1\rangle \text{ (with prob. } \langle\psi|\Pi_1|\psi\rangle) \end{cases}$$

What is the **expectation value** of z?

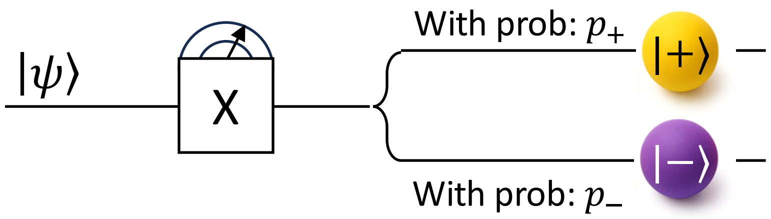
$$\mathbb{E}[z] = (+1) \cdot \Pr[z = +1] + (+1) \cdot \Pr[z = -1]$$

$$= \langle\psi|Z|\psi\rangle \equiv \langle z\rangle$$

- Measured values and probabilities are determined by the eigenvalues and eigenvectors of Z.
- In physics, such an operator $\sigma_Z = Z$ is called an **observable** to represent a physical quantity that can be measured.
- An **observable** must be Hermitian. (Why?)

Measuring in a Different Basis

Measurement along the **x basis** ("Hadamard basis"):



If we define a **random variable x**:

$$x = \begin{cases} +1 \text{ if } |\psi\rangle \text{ collapsed to } |+\rangle \text{ (with prob. } \langle\psi|\Pi_+|\psi\rangle) \\ -1 \text{ if } |\psi\rangle \text{ collapsed to } |-\rangle \text{ (with prob. } \langle\psi|\Pi_-|\psi\rangle) \end{cases}$$

The **expectation value** of x: $\langle x\rangle \equiv \langle\psi|X|\psi\rangle$

Example:

For the following $|\psi\rangle$, what are the probabilities p_+ and p_-?

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

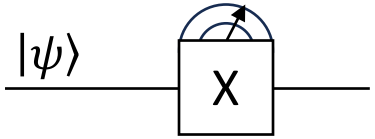
- $\langle+|\psi\rangle = \frac{1+i}{2}$
- $\langle-|\psi\rangle = \frac{1-i}{2}$

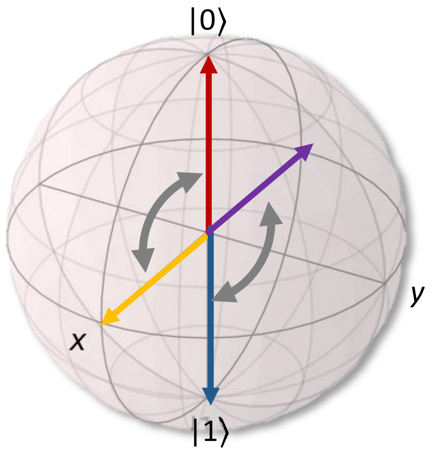
- $p_+ = \left|\frac{1+i}{2}\right|^2 = \frac{1}{2}$
- $p_- = \left|\frac{1-i}{2}\right|^2 = \frac{1}{2}$

How is this implemented physically?

Measuring in a Different Basis

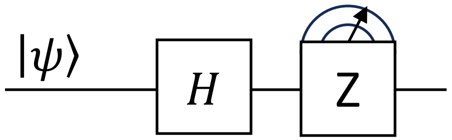
Measurement along the **x basis** (“Hadamard basis”):





Hadamard gate:
“Swaps x and z axes!”

Simulating x-basis measurement using quantum gates and standard-basis measurement?



- It produces the **same statistics** as an x-basis measurement.
- Does it produce the same post-measurement state?

Experimental implication: Standard-basis measurement + quantum gates are computationally universal.

Heisenberg Uncertainty Principle

The **uncertainty principle** states that both observables **cannot** yield definite (non-random) outcomes on the same state.

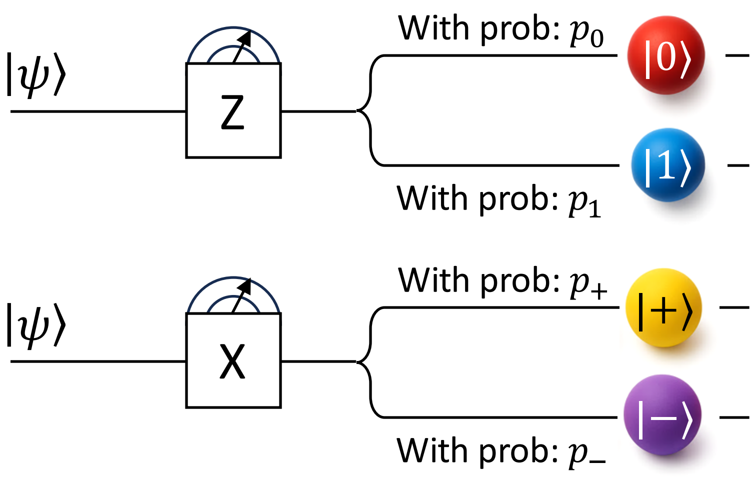
More formally, for two observables A, B and a given quantum state $|\psi\rangle$:

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle\psi|[A, B]|\psi\rangle|$$

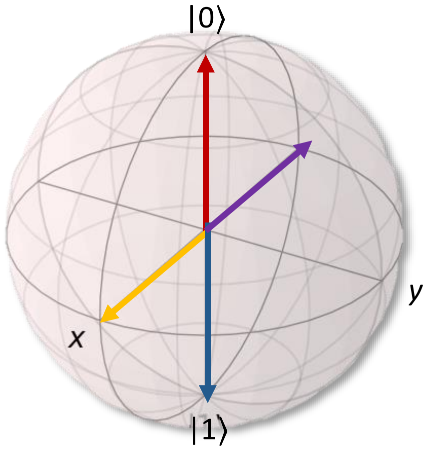
where ΔA and ΔB are the standard deviation of observables A and B, $[A, B] = AB - BA$ is their commutator.

Scenario #1:

Scenario #2:



If measuring $|\psi\rangle$ in standard basis yields a definite outcome, then its Hadamard basis measurement **cannot** have a definite outcome.



Non-commuting observables are called **incompatible**:
E.g., position and momentum of a particle cannot be simultaneously determined.

General Measurement: Born's Rule

Given an **observable** $O = \sum_n a_n E_n$, where $a_n \in \mathbb{R}$ and $E_n E_m = \delta_{nm} E_n, E_n = E_n^\dagger$

(real eigenvalues) (orthonormal projectors)

We can define the measurement by observable O acting on quantum state $|\psi\rangle$ to:

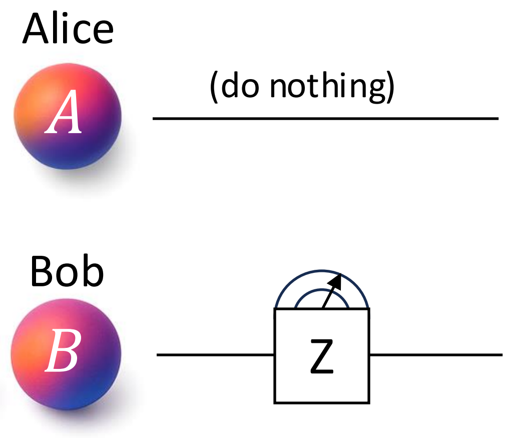
- Produce a (classical) **readout** value: a_n
- **Collapse** the quantum state to: $\frac{E_n|\psi\rangle}{\sqrt{\langle\psi|E_n|\psi\rangle}}$
- With **probability**: $\langle\psi|E_n|\psi\rangle$.

The **expectation value** of this measurement:

$$\langle O\rangle = \sum_n a_n \operatorname{Pr}[\text{readout} = a_n] = \sum_n a_n \langle\psi|E_n|\psi\rangle = \left\langle\psi\middle|\left(\sum_n a_n E_n\right)\middle|\psi\right\rangle = \langle\psi|O|\psi\rangle$$

Partial Measurement

Scenario #1: Suppose $|\psi_{AB}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.
What if only Bob measures his qubit?

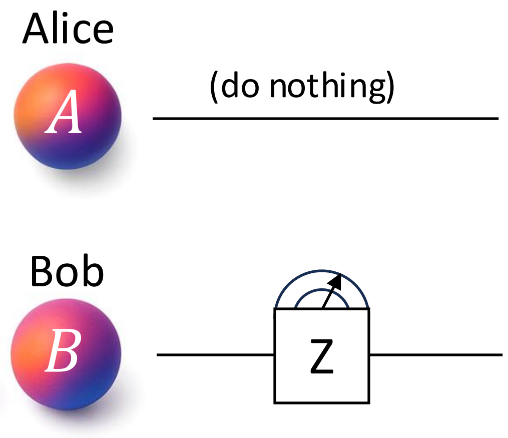


Derive on board:
What is the **probability distribution** of his measurement outcomes?

$$|\psi_{AB}\rangle \text{ collapses to: } \begin{cases} |+\rangle \otimes |0\rangle & \text{if Bob readout is } +1 \text{ (with prob. } 1/2 \text{)} \\ |+\rangle \otimes |1\rangle & \text{if Bob readout is } -1 \text{ (with prob. } 1/2 \text{)} \end{cases}$$

Partial Measurement

Scenario #2: Suppose $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle$



Derive on board:

What is the **probability distribution** of his measurement outcomes?

$|\psi_{AB}\rangle$ collapses to: $\begin{cases} |0\rangle \otimes |0\rangle & \text{if Bob readout is } +1 \text{ (with prob. } 1/2 \text{)} \\ |+\rangle \otimes |1\rangle & \text{if Bob readout is } -1 \text{ (with prob. } 1/2 \text{)} \end{cases}$

More formally, we can re-write:

$$|\psi\rangle = \sum_{j,k} \alpha_{jk}|jk\rangle = \sum_k \left(\sum_j \alpha_{jk}|j\rangle\right) \otimes |k\rangle$$

"All amplitude consistent with k."

Let $\beta_k = \sqrt{\sum_j |\alpha_{jk}|^2}$, then we have

$$|\psi\rangle = \sum_k \beta_k \left(\sum_j \frac{\alpha_{jk}}{\beta_k}|j\rangle\right) \otimes |k\rangle$$

With probability β_k^2, the state collapses to:

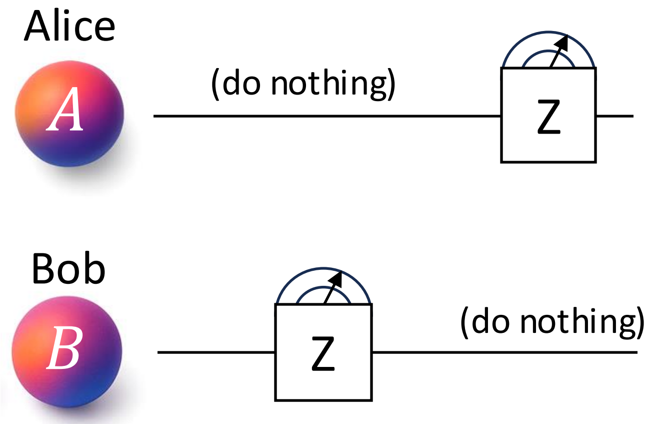
$$\left(\sum_j \frac{\alpha_{jk}}{\beta_k}|j\rangle\right) \otimes |k\rangle$$

Qubit A Qubit B

Measuring One Qubit at a Time

Scenario #3: Suppose $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle$

Bob measures his qubit first, then Alice measures hers.

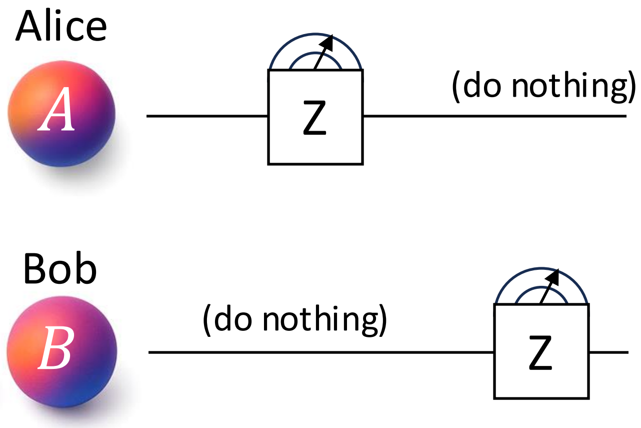


Bob's Statistics	States after Bob's measurement	Alice's Statistics	State after Alice's measurement
$\Pr[\|0\rangle_B] = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$	$\|00\rangle$	$\Pr[\|0\rangle_A \| \|0\rangle_B] = 1$	$\|00\rangle$
		$\Pr[\|1\rangle_A \| \|0\rangle_B] = 0$	$-$
$\Pr[\|1\rangle_B] = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$	$\frac{1}{\sqrt{2}}\|01\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\Pr[\|0\rangle_A \| \|1\rangle_B] = \frac{1}{2}$	$\|01\rangle$
		$\Pr[\|1\rangle_A \| \|1\rangle_B] = \frac{1}{2}$	$\|11\rangle$

Measuring One Qubit at a Time

Scenario #4: Suppose $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle$.

Had Alice measured first: What is the probability distribution and collapsed state?



Alice's Statistics	States after Alice's measurement	Bob's Statistics	State after Bob's measurement
$\Pr[\|0\rangle_A] = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4}$	$\frac{2}{\sqrt{6}}\|00\rangle + \frac{1}{\sqrt{3}}\|01\rangle$	$\Pr[\|0\rangle_B \| \|0\rangle_A] = \frac{2}{3}$	$\|00\rangle$
		$\Pr[\|1\rangle_B \| \|0\rangle_A] = \frac{1}{3}$	$\|01\rangle$
$\Pr[\|1\rangle_A] = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\|11\rangle$	$\Pr[\|0\rangle_B \| \|1\rangle_A] = 0$	–
		$\Pr[\|1\rangle_B \| \|1\rangle_A] = 1$	$\|11\rangle$

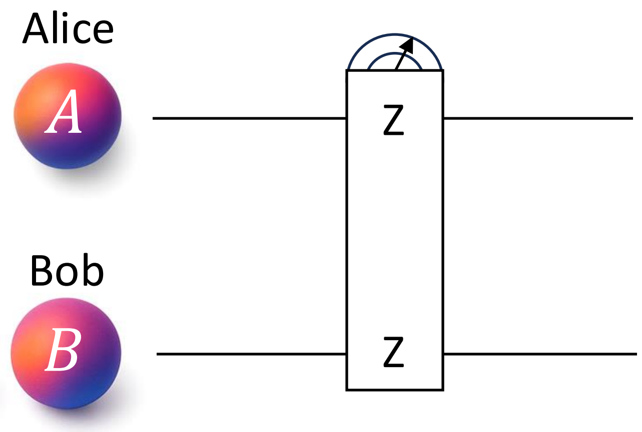
Joint Measurement

Measuring two qubits **jointly** using observable: $O = Z \otimes Z = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$

$$= (+1)\underbrace{(|00\rangle\langle 00| + |11\rangle\langle 11|)}_{E_+\text{: even parity}} + (-1)\underbrace{(|01\rangle\langle 01| + |10\rangle\langle 10|)}_{E_-\text{: odd parity}}$$

E_+: even parity

E_-: odd parity



"Measuring **parity** of A and B."

Measuring $Z \otimes Z$ collapses quantum state $|\psi\rangle$ to:

- $\frac{E_+|\psi\rangle}{\sqrt{\langle\psi|E_+|\psi\rangle}}$ with probability: $\langle\psi|E_+|\psi\rangle$.
- $\frac{E_-|\psi\rangle}{\sqrt{\langle\psi|E_-|\psi\rangle}}$ with probability: $\langle\psi|E_-|\psi\rangle$.

Example: $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle$ collapses to:

(even, odd, even)

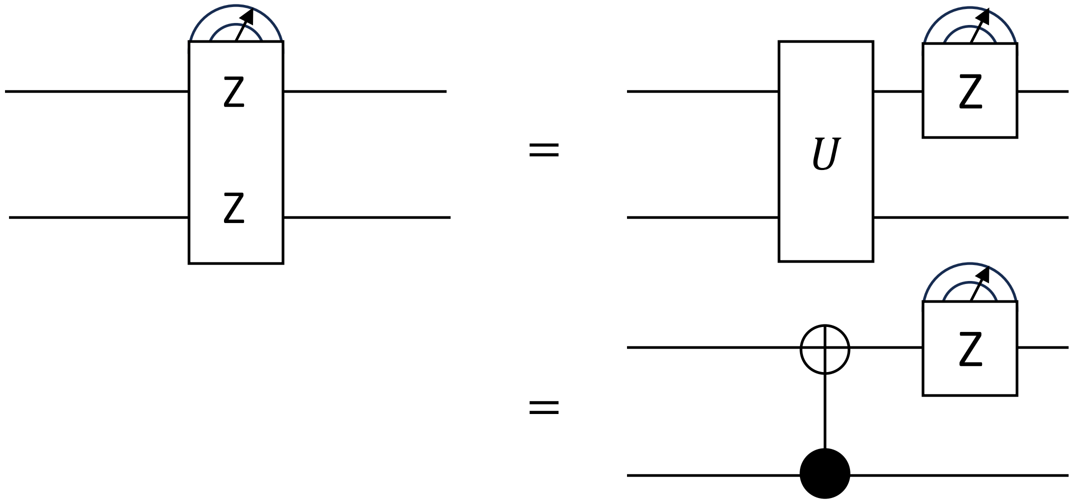
$$\begin{cases} \frac{2}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|11\rangle & \text{(with prob. } \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4}\text{)} \\ |01\rangle & \text{(with prob. } \left(\frac{1}{2}\right)^2 = \frac{1}{4}\text{)} \end{cases}$$

Joint Measurement

Simulating joint measurement using quantum gates and (single-qubit) standard-basis measurement?

Getting the measurement statistics of the "parity" basis (e.g., $\langle Z \otimes Z \rangle$) requires:

- Obtaining one bit of information, and
- Entangling gates.



Derive on board: Finding U such that

$$O = Z \otimes Z = U^{\dagger}(Z \otimes I)U$$

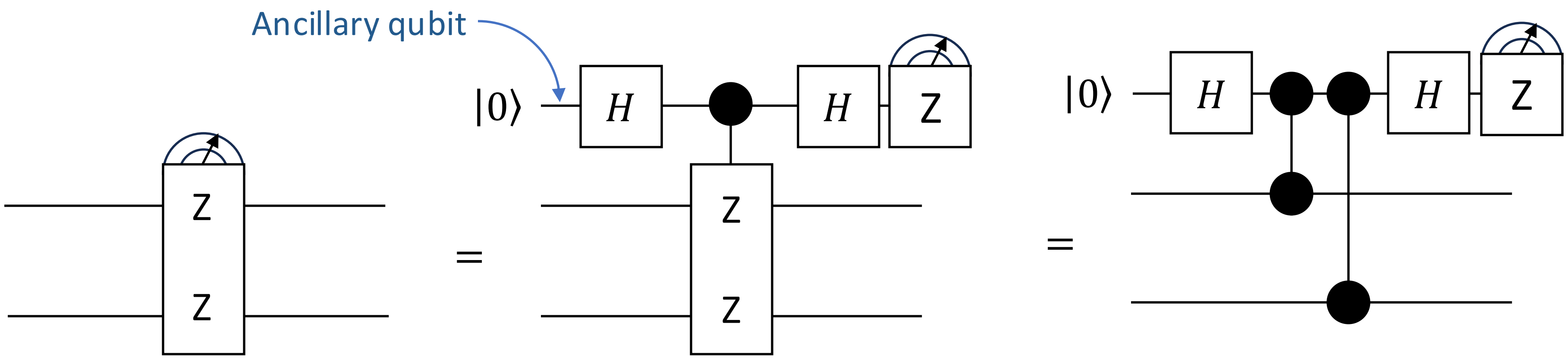
It samples from the correct distribution,
but does it get the correct post-meas. states?

Joint Measurement

Simulating joint measurement using quantum gates and (single-qubit) standard-basis measurement?

Getting the measurement statistics **and** post meas. state of the "parity" basis (e.g., $\langle Z \otimes Z \rangle$) requires:

- Obtaining one bit of information, and
- Entangling gates, and
- Collapsing qubits to their appropriate superposition.



<u>Derive on board</u>: What're the effective projectors on the two (data) qubits?