

Quantum Teleportation



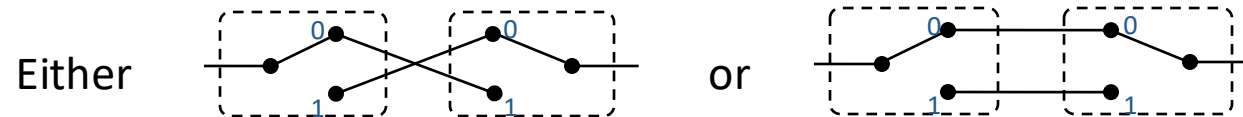
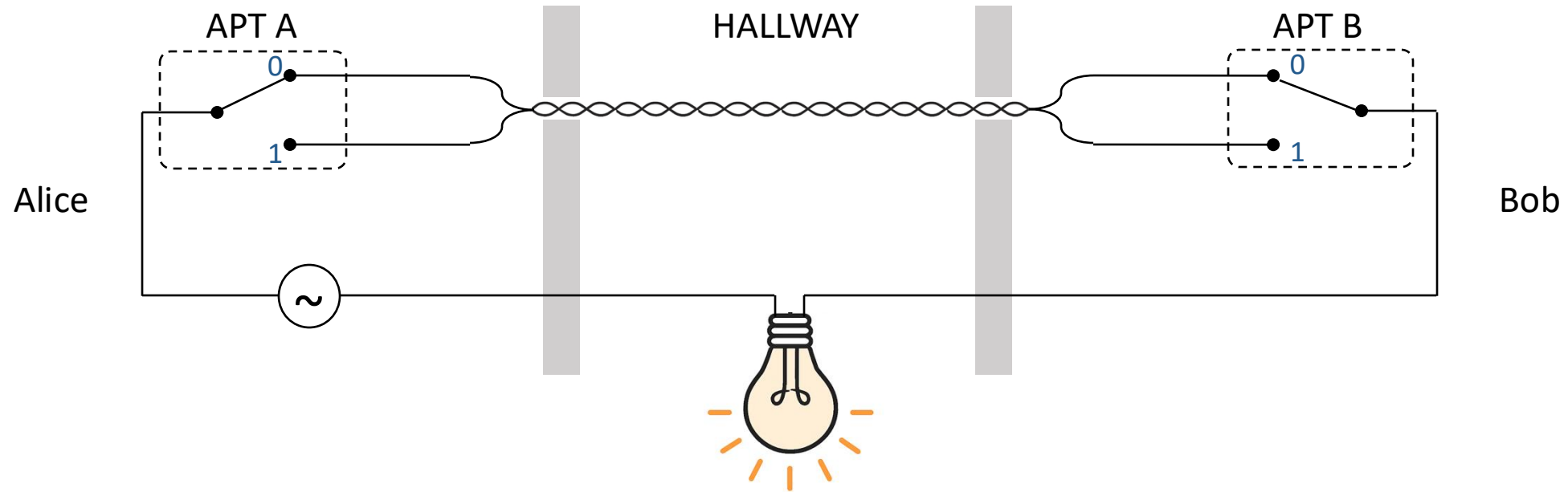
CPSC 4470/5470

Introduction to Quantum Computing

Instructor: Prof. **Yongshan Ding**

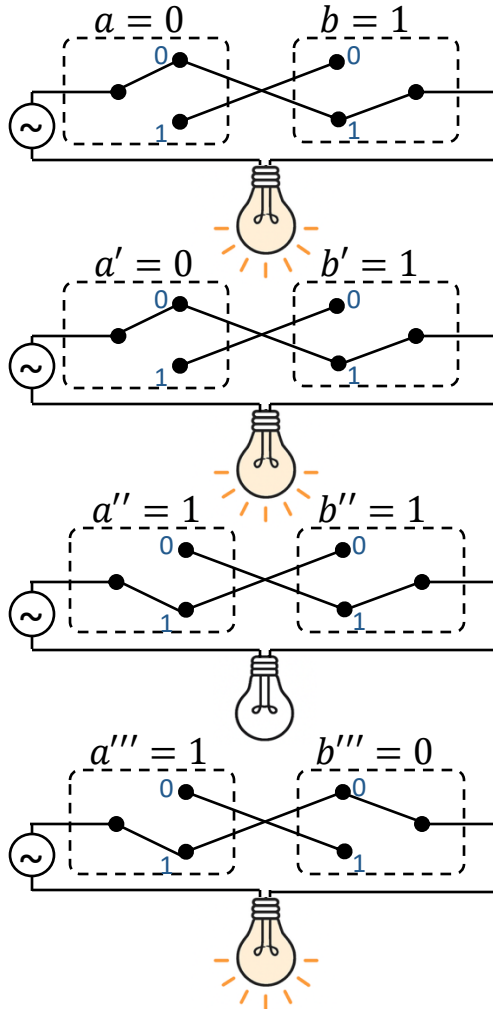
Computer Science, Applied Physics, Yale Quantum Institute

Information Swapping



Can Alice and Bob exchange their information via the light bulb?

Implementing Swap with XORs



$\oplus$: XOR

$$a' = a = 0$$

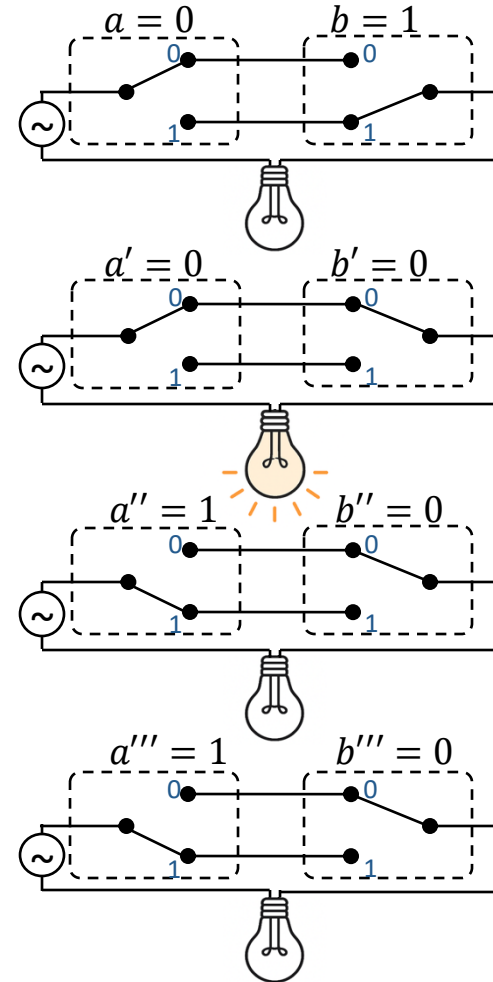
$$b' = a \oplus b = 1$$

$$a'' = a' \oplus b' = 1$$

$$b'' = b' = 1$$

$$a''' = a'' = 1$$

$$b''' = a'' \oplus b'' = 0$$



$\odot$: XNOR

$$a' = a = 0$$

$$b' = a \odot b = 0$$

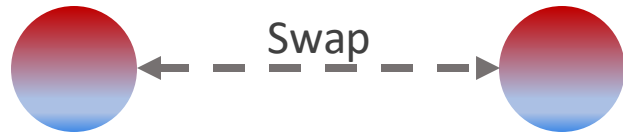
$$a'' = a' \odot b' = 1$$

$$b'' = b' = 0$$

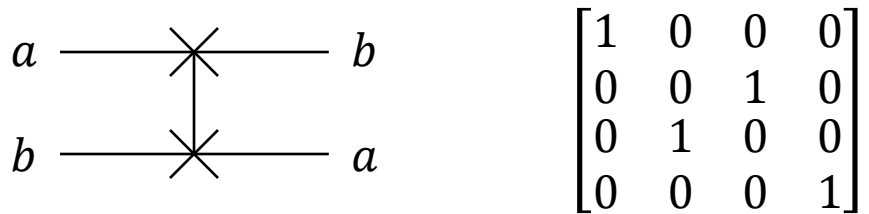
$$a''' = a'' = 1$$

$$b''' = a'' \odot b'' = 0$$

Quantum Information Swapping

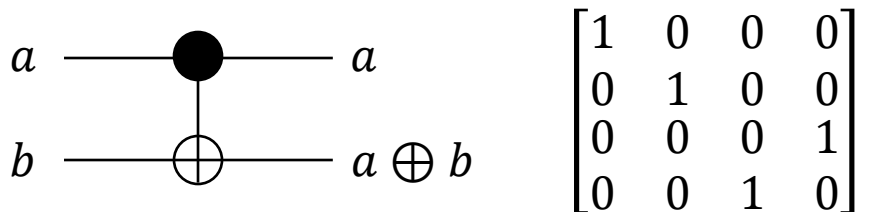


A quantum **swap gate** exchanges the amplitudes on $|01\rangle$ and $|10\rangle$.



$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \alpha_{11} \end{bmatrix}$$

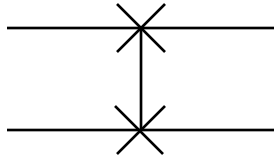
CX(0,1) gate exchanges the amplitudes on $|10\rangle$ and $|11\rangle$.



$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{11} \\ \alpha_{10} \end{bmatrix}$$

Implementing Swap with XORs

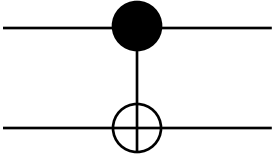
Swap:



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Swap gate exchanges the amplitudes on $|01\rangle$ and $|10\rangle$.

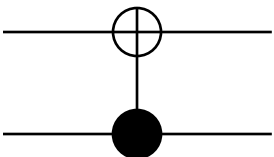
CX(0,1)



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

CX(0,1) gate exchanges the amplitudes on $|10\rangle$ and $|11\rangle$.

CX(1,0)



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

CX(1,0) gate exchanges the amplitudes on $|01\rangle$ and $|11\rangle$.

$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \alpha_{11} \end{bmatrix}$$

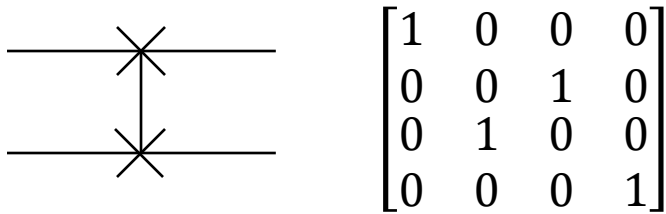
$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{11} \\ \alpha_{10} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_{00} \\ \alpha_{11} \\ \alpha_{10} \\ \alpha_{01} \end{bmatrix}$$

How to implement a Swap using CX gates?

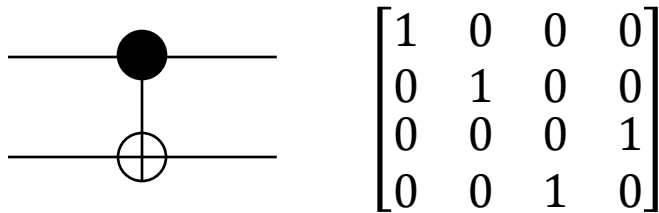
Implementing Swap with XORs

Swap:



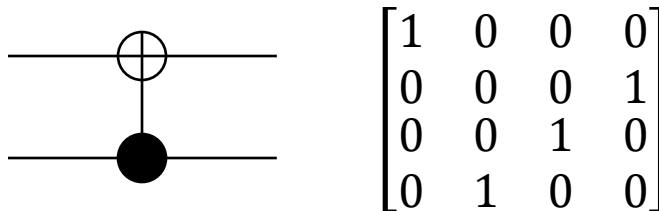
Swap gate exchanges the amplitudes on $|01\rangle$ and $|10\rangle$.

CX(0,1)



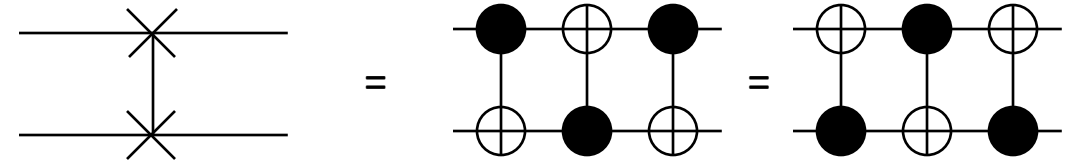
CX(0,1) gate exchanges the amplitudes on $|10\rangle$ and $|11\rangle$.

CX(1,0)

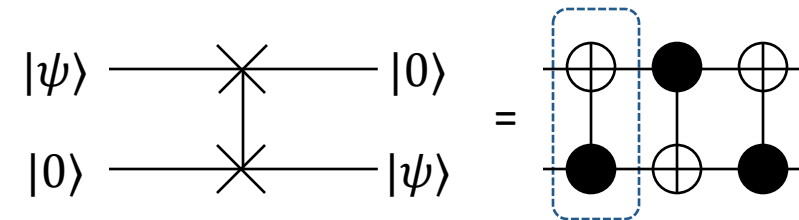


CX(1,0) gate exchanges the amplitudes on $|01\rangle$ and $|11\rangle$.

How to implement Swap using CX gates?

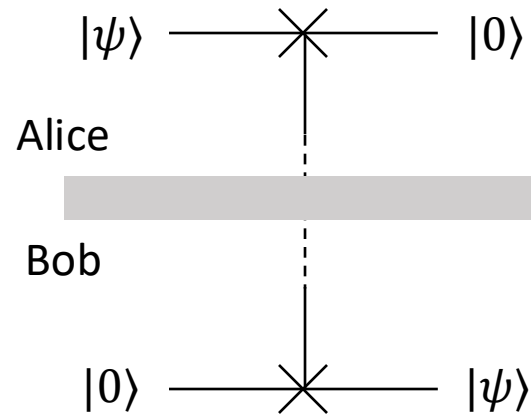


What if we assume one of the qubits is $|0\rangle$?



Implementing Remote Swap

Remote Swap:

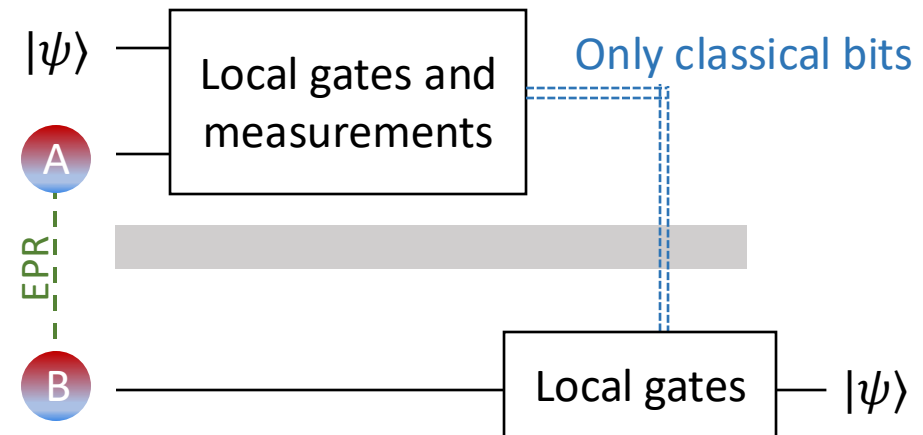


Across the barrier:

- Direct two-qubit gates are **not allowed** between the qubits.
- Communicating classical bits are **allowed**.
- Prior entanglement is also **allowed**.

With the help of entanglement:

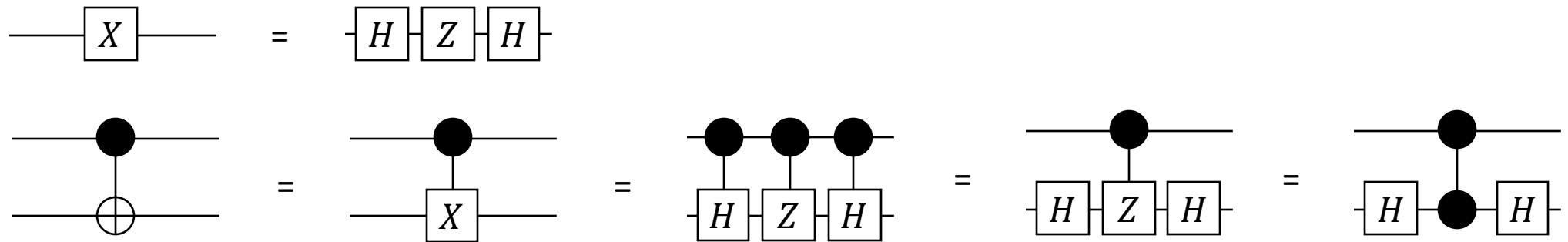
$$\begin{array}{c} \text{A} \\ \vdots \\ \text{B} \end{array} \text{EPR} = |\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$



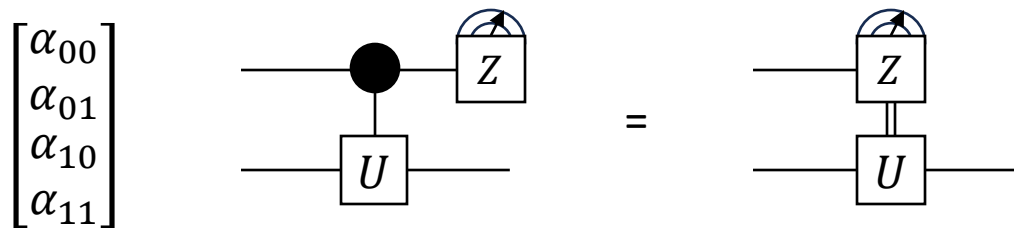
Let's see how we can accomplish this next.

Some Useful Facts about Quantum Circuits

Conjugation Relations: $X = HZH, Z = HXH$



Measurement and Control: $|0\rangle(\alpha_{00}|0\rangle + \alpha_{01}|1\rangle) + |1\rangle U(\alpha_{10}|0\rangle + \alpha_{11}|1\rangle)$

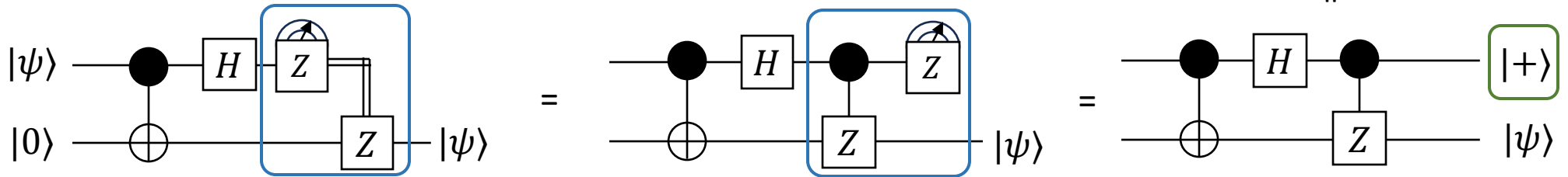


If measurement outcome is 1, then perform U gate.

Quantum controlled gate

Classical conditional gate

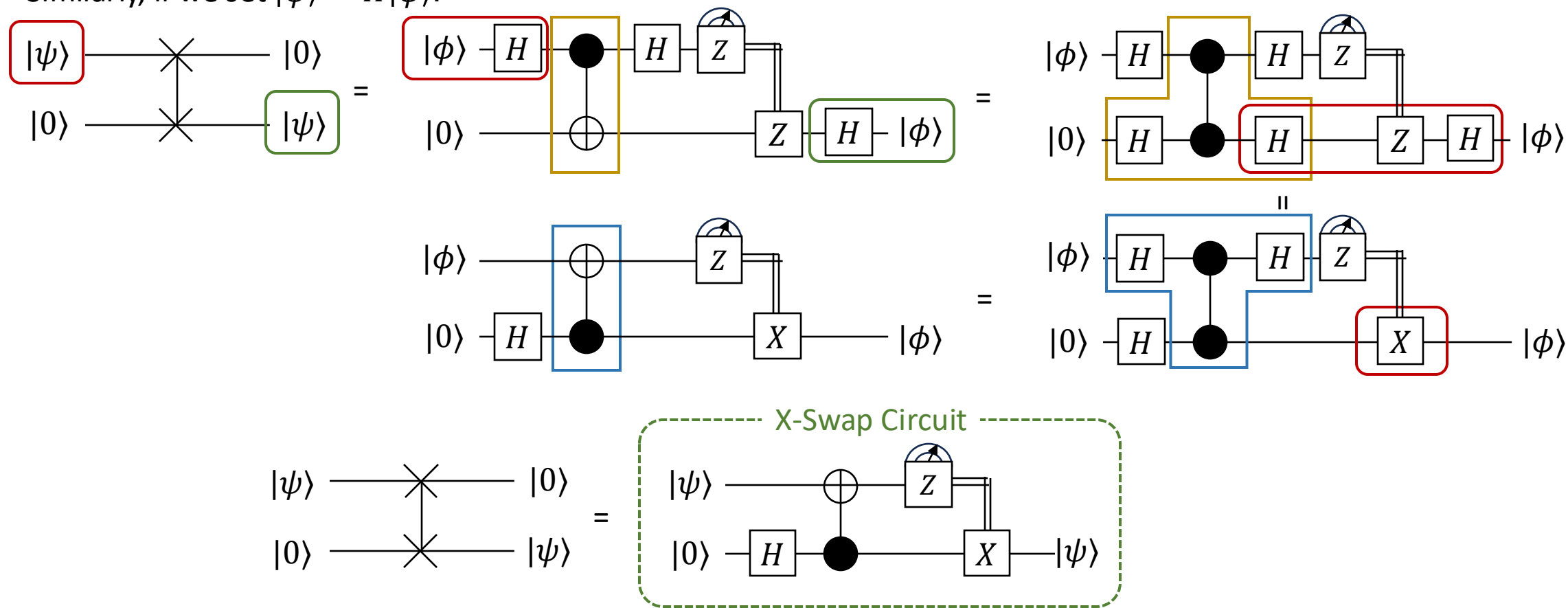
One-Bit Teleportation



X-Swap Circuit

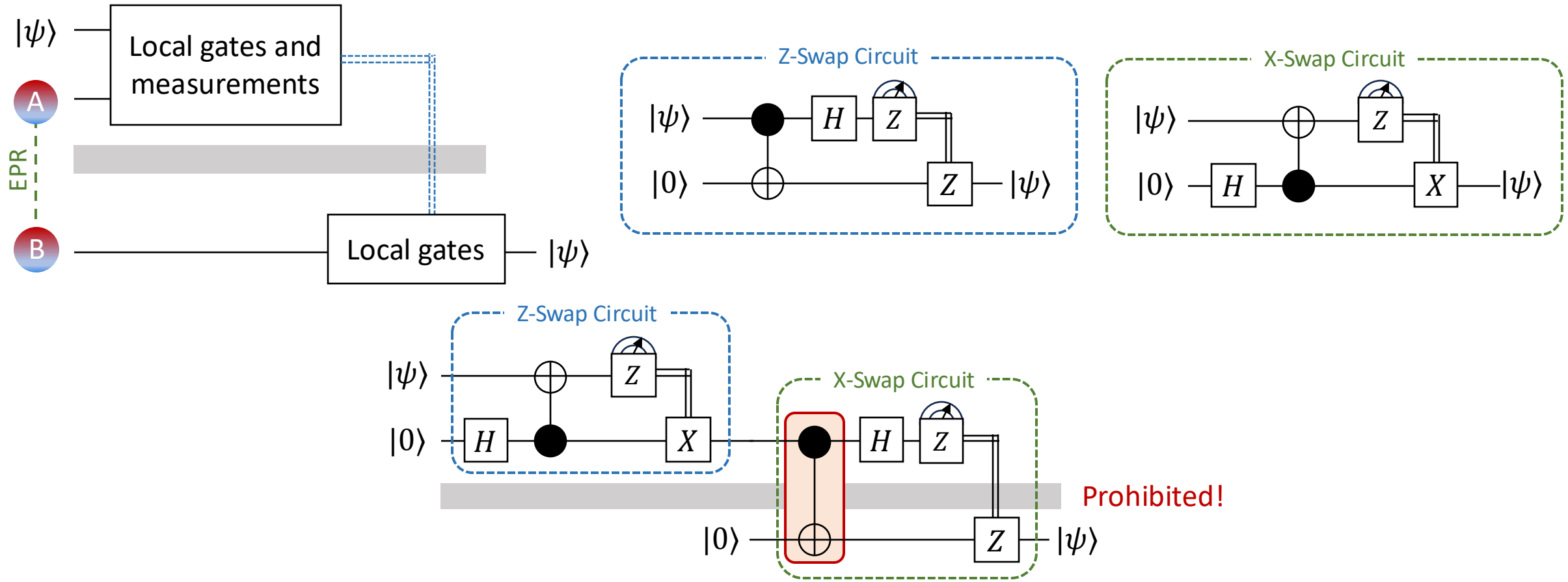
One-Bit Teleportation

Similarly, if we set $|\psi\rangle = H|\phi\rangle$.



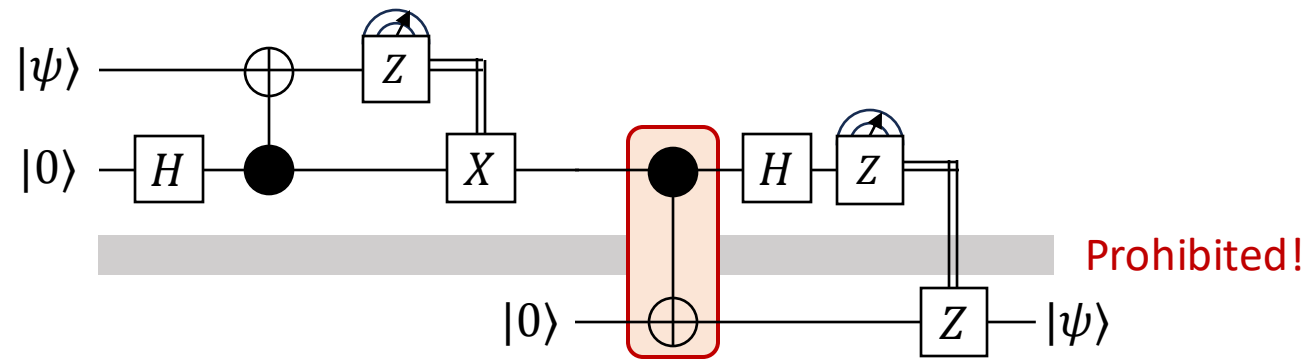
(Two-Bit) Teleportation Circuit

Putting it all together

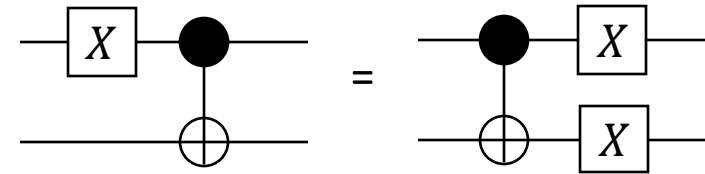


(Two-Bit) Teleportation Circuit

Putting it all together



More Circuit Equivalences:

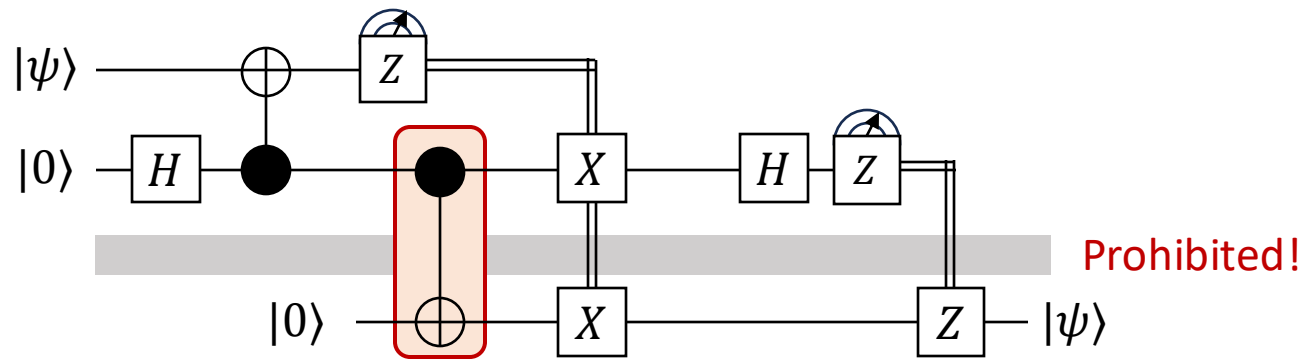


"Commute CX to the left."

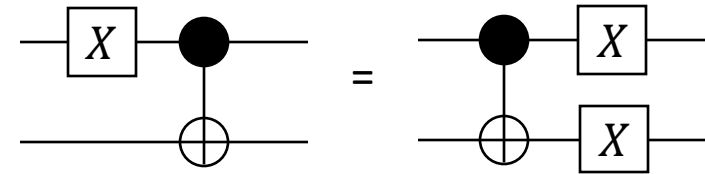
$$CX_{1,2}(X_1 \otimes I_2) = (X_1 \otimes X_2)CX_{1,2}$$

(Two-Bit) Teleportation Circuit

Putting it all together

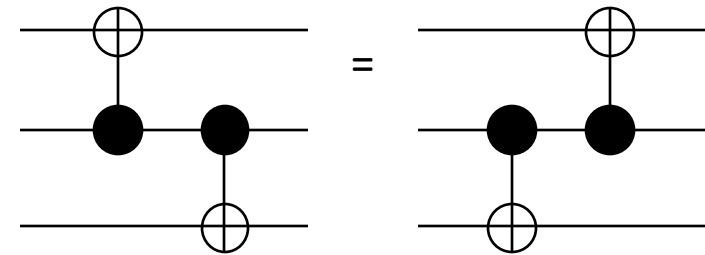


More Circuit Equivalences:



"Commute CX to the left."

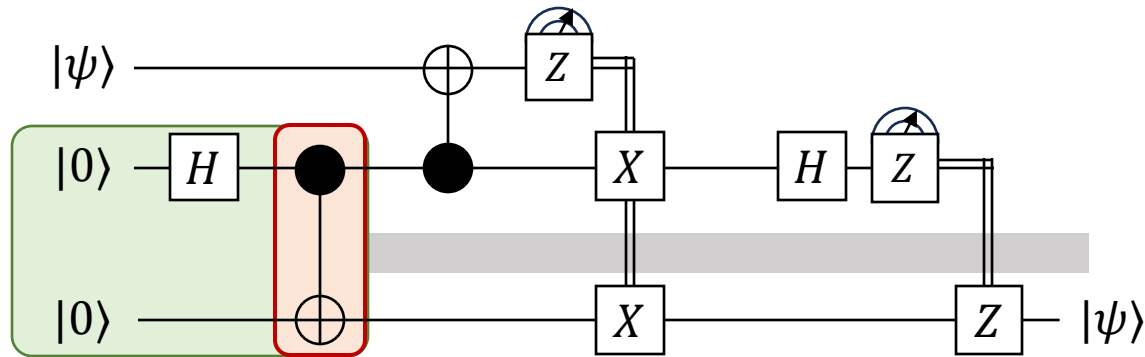
$$CX_{1,2}(X_1 \otimes I_2) = (X_1 \otimes X_2)CX_{1,2}$$



"Controls commute with each other."

(Two-Bit) Teleportation Circuit

Putting it all together



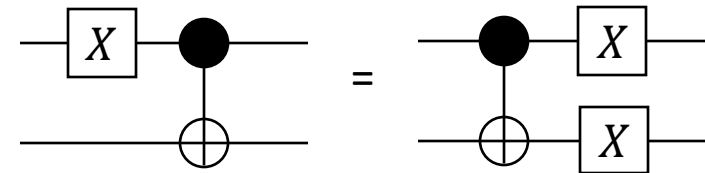
Bell State: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

(generate ahead of time)

This circuit works for **teleporting any arbitrary quantum state!**

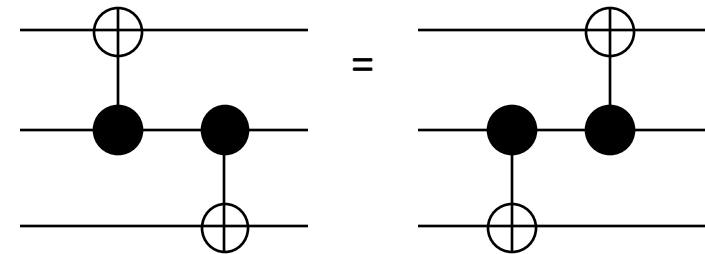
- Entanglement (Bell/EPR pair) shared prior to teleportation.
- No quantum gates between Alice and Bob.
- Classical communication (of 2 classical bits) is needed.
- No violation of the no-cloning theorem.
- No qubit is moved, only information.
- No faster-than-light communication or teleportation.

More Circuit Equivalences:



“Commute CX to the left.”

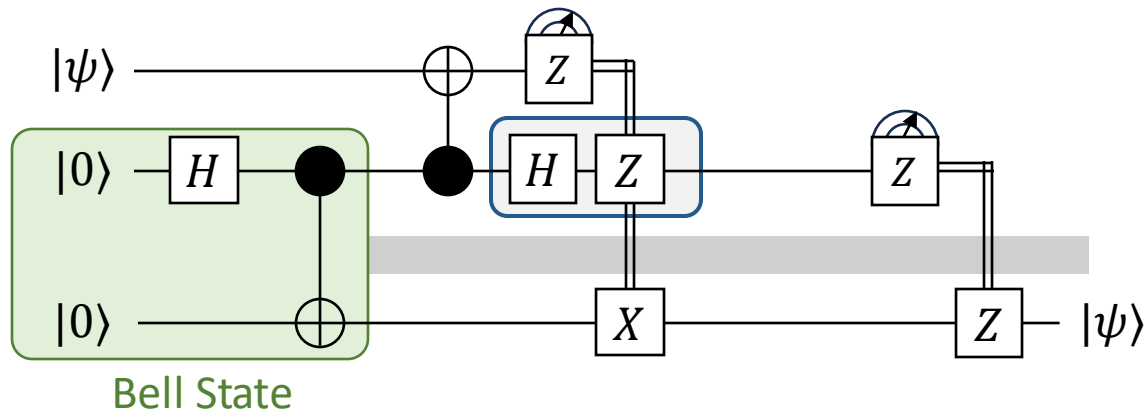
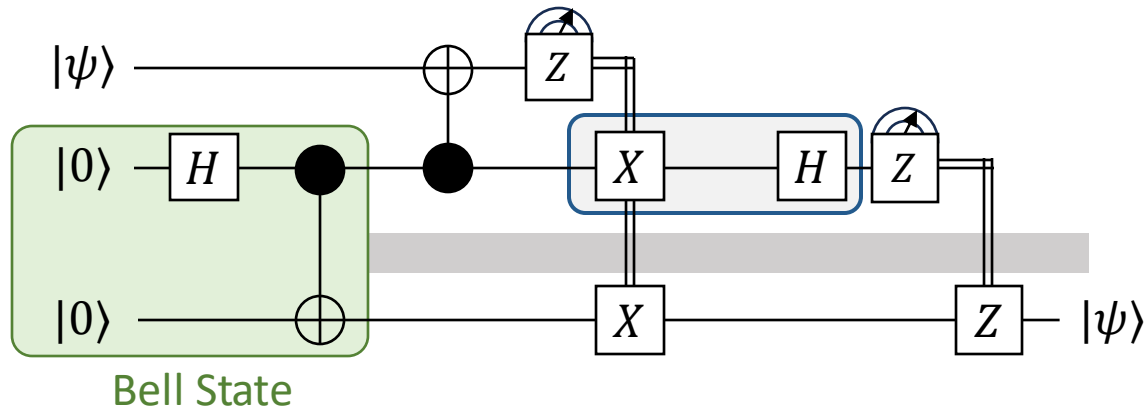
$$CX_{1,2}(X_1 \otimes I_2) = (X_1 \otimes X_2)CX_{1,2}$$



“Controls commute with each other.”

(Two-Bit) Teleportation Circuit

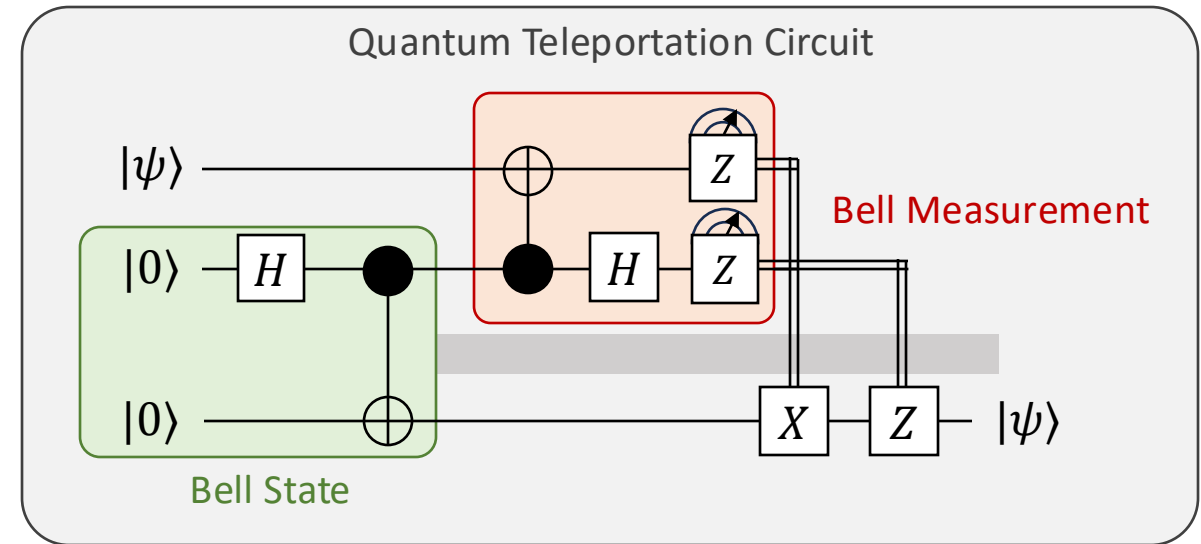
Putting it all together



More Circuit Equivalences:

$$\text{---} [X] [H] \text{---} = \text{---} [H] [Z] \text{---} \quad X = HZH$$

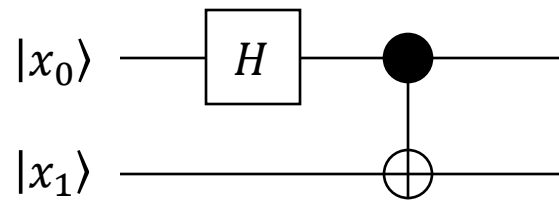
$$\text{---} [Z] [Z] \text{---} = \text{---} [Z] \text{---}$$



Bell Measurement: “Entanglement Detector”

Defining a new *orthonormal basis* for two qubits:

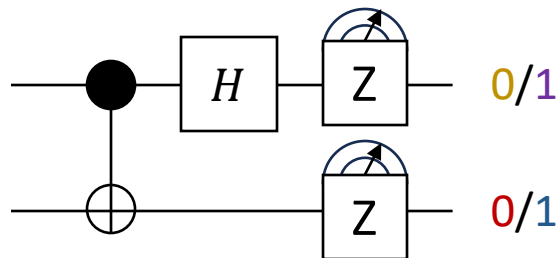
Bell state preparation:



$$\left\{ \begin{array}{ll} \text{If } |x_0x_1\rangle = |00\rangle, |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \text{Even parity, in phase} \\ \text{If } |x_0x_1\rangle = |10\rangle, |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) & \text{Even parity, out of phase} \\ \text{If } |x_0x_1\rangle = |01\rangle, |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) & \text{Odd parity, in phase} \\ \text{If } |x_0x_1\rangle = |11\rangle, |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) & \text{Odd parity, out of phase} \end{array} \right.$$

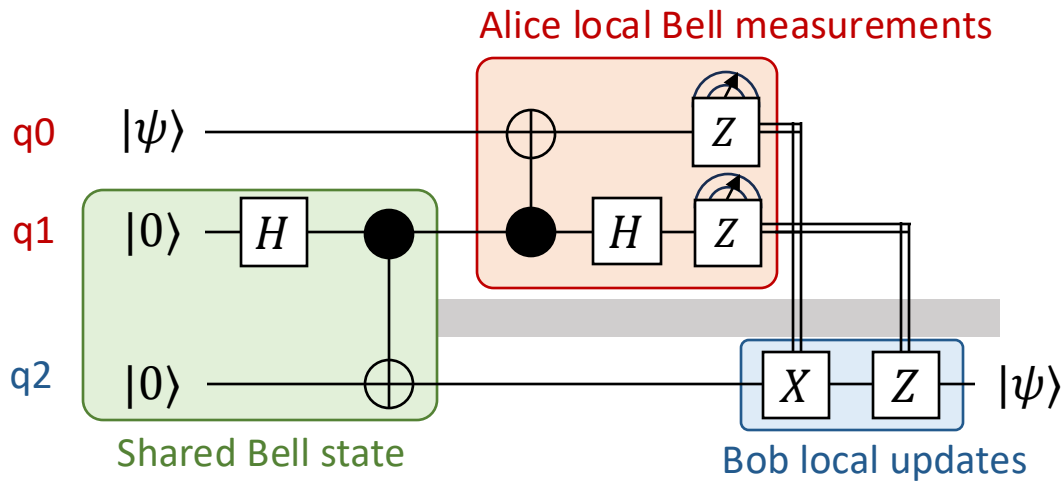
Four **Bell basis states**: maximally entangled

Bell basis measurement:



- Instead of asking “what are the bits individually?”
- We ask “which entanglement pattern do they share?”

Understanding Teleportation



Bell Basis:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

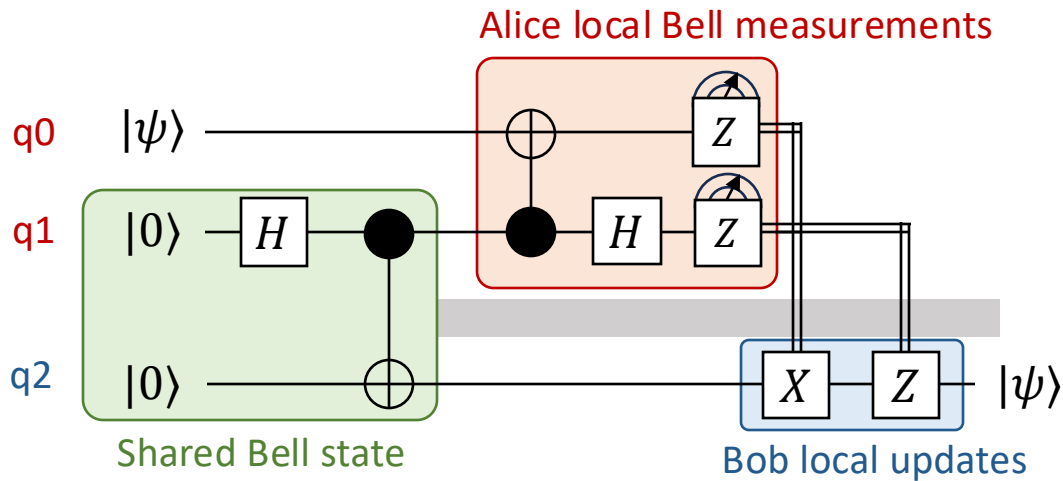
Start of teleportation:

$$(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Derive on board:

- What is Alice's measurement distribution?
- What is the state before Bob's local updates?

Protocol Implementation



Alice's quantum register: {q0, q1}

Bob's quantum register: {q2}

Alice's classical register: {c0, c1}

Bob's classical register: {c0, c1, c2}

Main program for the teleportation protocol:

```
def teleportation():  
    qc = QuantumCircuit(3, 3)  
  
    # EPR generation on q1, q2  
    qc.h(1)  
    qc.cx(1, 2)
```

SHARE entangled qubits between Alice and Bob

```
# Send q0, q1 to Alice and q2 to Bob  
alice_ids = [0, 1]  
bob_ids = [2]  
(alice_qreg, alice_creg) = qc.allocate(alice_ids, 0)  
(bob_qreg, bob_creg) = qc.allocate(bob_ids, 0)  
  
# Build subcircuits for Alice and Bob  
alice_subcircuit = QuantumCircuit(alice_qreg, alice_creg)  
share_creg = bob_creg + alice_creg  
shared_ids = bob_ids + alice_ids  
bob_subcircuit = QuantumCircuit(bob_qreg, share_creg)
```

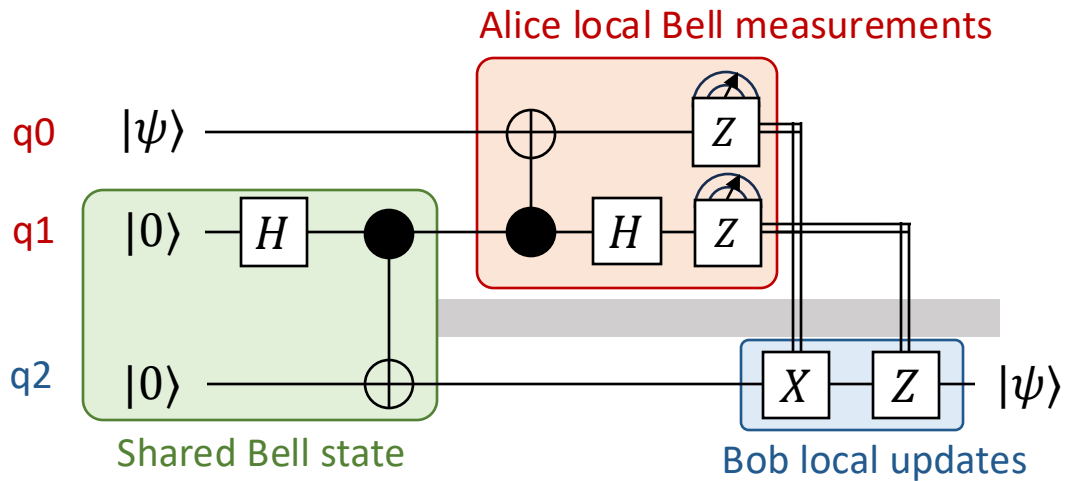
WAIT for signal to start teleportation

```
# Alice performs her local operations  
alice_locals(alice_subcircuit)  
qc._append(alice_subcircuit, alice_ids, alice_ids)
```

WAIT for Alice's measurement signal

```
# Bob performs his local update  
bob_updates(bob_subcircuit)  
qc._append(bob_subcircuit, bob_ids, shared_ids)  
  
return qc
```

Protocol Implementation



Classical conditional gate:

```
# conditional gate: if all 'cbits' are 1, then perform 'gate' on 'qubits'
def conditional(self, cbits, gate, qubits):
    assert(len(cbits) > 0)
    assert(isinstance(gate, str))
    assert(gate in self.gateset)
    cond = '{}_if'.format(gate)
    conditionalGate = Gate(cond, len(qubits), None)
    self._append(conditionalGate, qubits, cbits)
    return
```

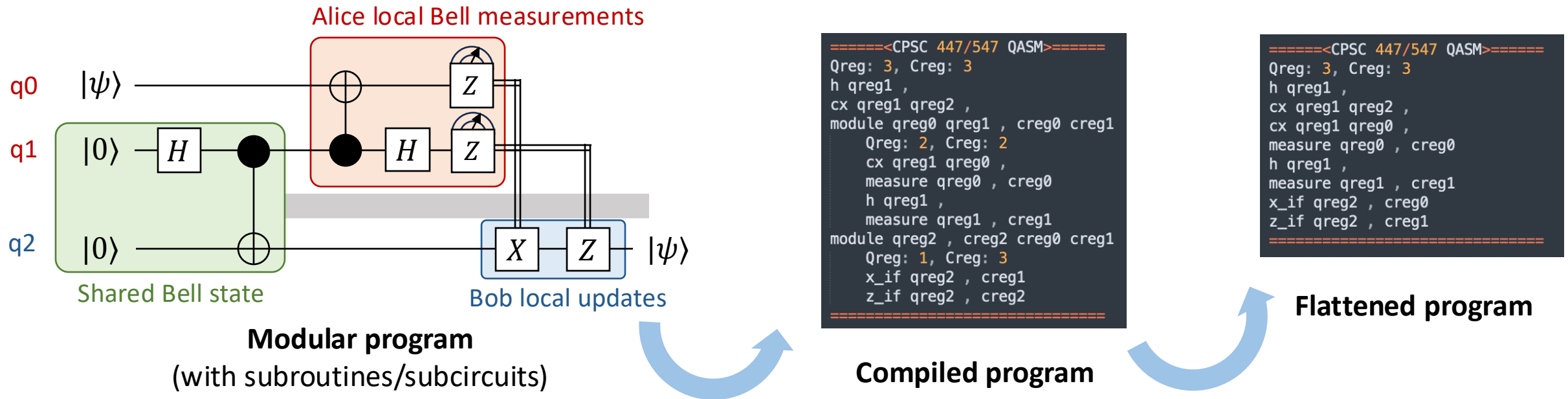
Alice's local program:

```
def alice_locals(alice_qc):
    # qreg: [0, 1] = [q0, q1]
    # creg: [0, 1] = [c0, c1]
    alice_qc.cx(1, 0)
    alice_qc.measure([0], [0])
    alice_qc.h(1)
    alice_qc.measure([1], [1])
```

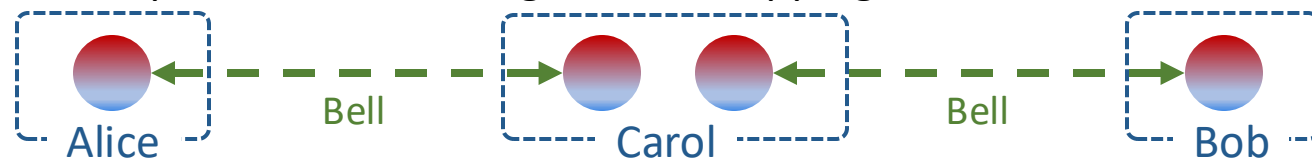
Bob's local program:

```
def bob_updates(bob_qc):
    # qreg: [0] = [q2]
    # creg: [0, 1, 2] = [c2, c0, c1]
    bob_qc.conditional([1], 'x', [0])
    bob_qc.conditional([2], 'z', [0])
```

Protocol Implementation

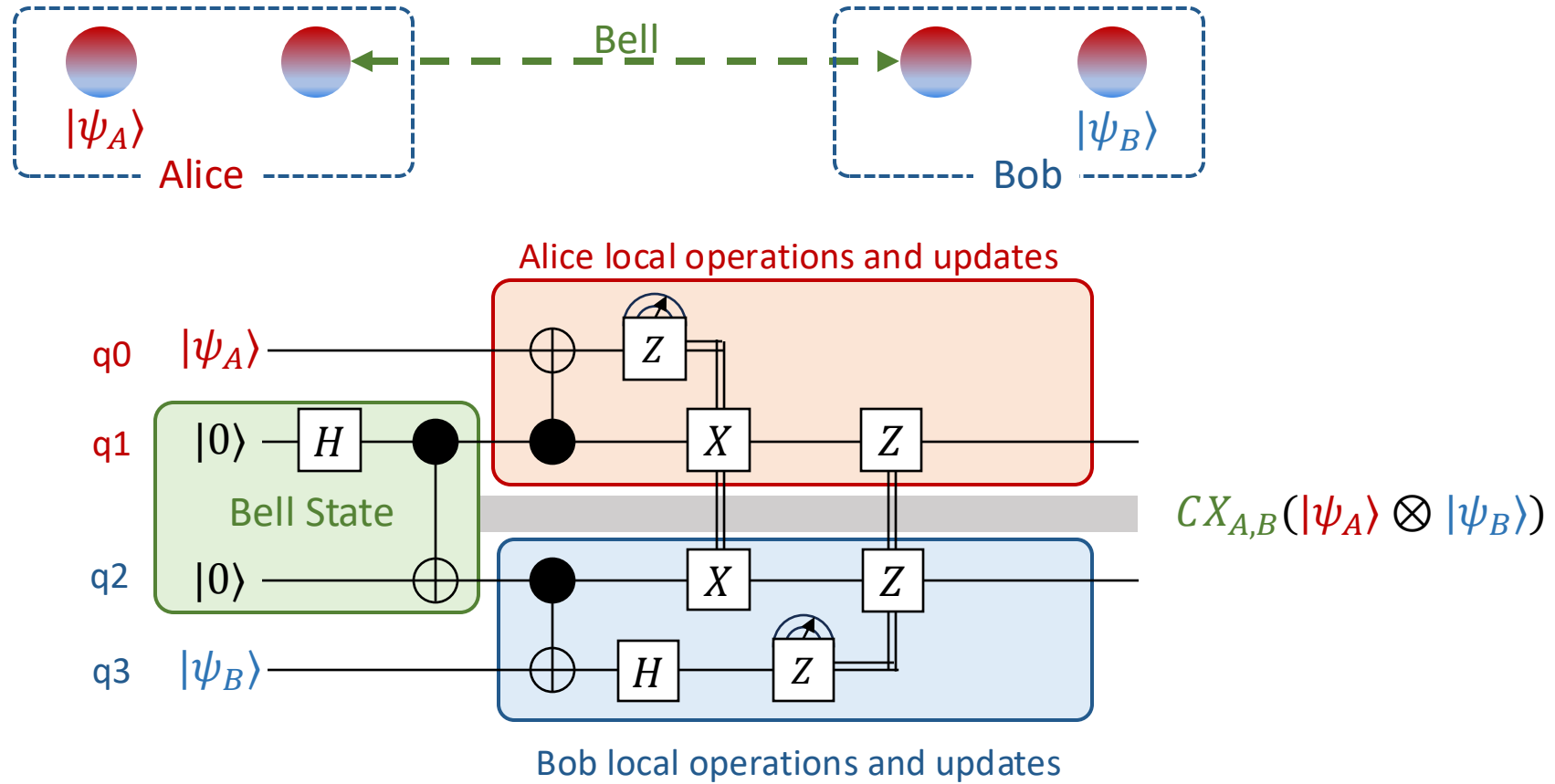


Pre-generated Bell State: How do Alice and Bob share entanglement in the first place, if they are far apart?
 Can Alice and Bob share entanglement, despite never directly interacted with each other?
Yes. This is your homework problem on “entanglement swapping”.



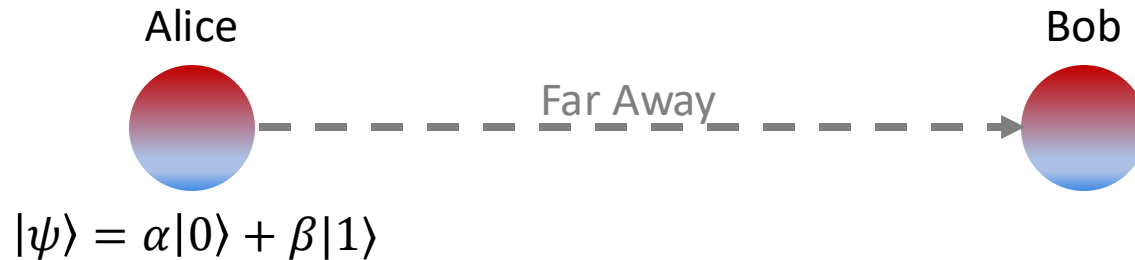
Teleporting Gates

Distributed Quantum Computing



Entanglement as a Resource

Quantum Teleportation: Quantum information swapping at a distance.



Direct quantum communication:

Physically sending the qubit $|\psi\rangle$ from Alice to Bob: e.g., as a photon.

Classical communication without entanglement:

Even for a known $|\psi\rangle$, we need classical communication of 2 complex numbers to tell Bob what is $|\psi\rangle$.

Classical communication with entanglement:

Classical communication of 2 classical bits and local gates by Alice and Bob.

Entanglement as a resource:

- Teleportation.
- Error-correcting code.
- Distributed computing.
- Certifiable randomness.
- Secure communication.
- Precise quantum sensors.
- Etc.