Lecture 3 CPSC 447/547 - Intro to QC

From Bits to Qubits (in computation)

Outline.

- Boolean Circuits
- Reversible circuits
- Randomized circuits
- Quantum circuits

Recall from the first lecture:

Information

	Bit	Qubit	Random bit
Possible State	0 or 1	$\alpha_0\|0\rangle + \alpha_1\|1\rangle$ $\alpha_0, \alpha_1 \in \mathbb{C}$ $\|\alpha_0\|^2 + \|\alpha_1\|^2 = 1$	p_0 "0" + p_1 "1" $p_0, p_1 \in [0,1]$ $p_0 + p_1 = 1$.

Multiple bits (e.g. 3 bits)

Joint State 8 possibilities:	000 or 001 or 010 or ⋮ or 111	$\alpha_{000}\|000\rangle + \alpha_{001}\|001\rangle + \alpha_{010}\|010\rangle + \cdots + \alpha_{111}\|111\rangle$ $\begin{bmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \vdots \\ \alpha_{111} \end{bmatrix}$	p_{000} "000" + p_{001} "001" + p_{010} "010" + ⋯ + p_{111} "111" $\begin{bmatrix} p_{000} \\ p_{001} \\ p_{010} \\ \vdots \\ p_{111} \end{bmatrix}$

State vector

Computation

(Deterministic, reversible, randomized, quantum)

Computational tasks can be modeled as transitions of states.

Or mathematically as Boolean Functions.

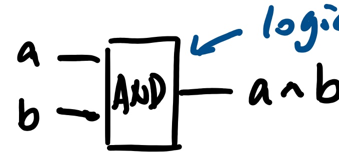
Example: AND function: $\{0,1\}^2 \to \{0,1\}$ maps 2 bits to 1 bit.

$$AND(a,b) = a \wedge b$$

Truth table to represent all transitions from input state to output state:

input		output
a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

We can also conveniently use a circuit diagram for this function:



logic gate

This way, we can represent very complex computational tasks as sequence of logic gates.

Proposition: Any Boolean function $f: \{0,1\}^m \to \{0,1\}^n$ can be computed by a Boolean circuit that consists of AND, OR, and NOT gates. "Universality".

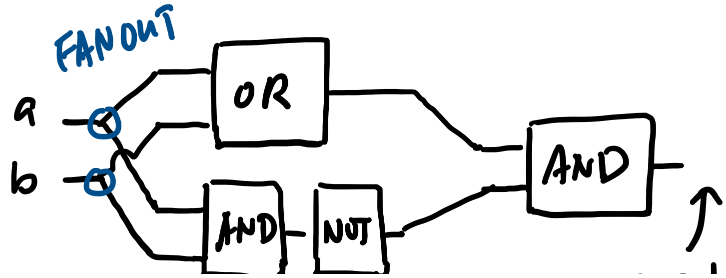
Example: XOR function: $XOR(a,b) = a \oplus b$

Truth table

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

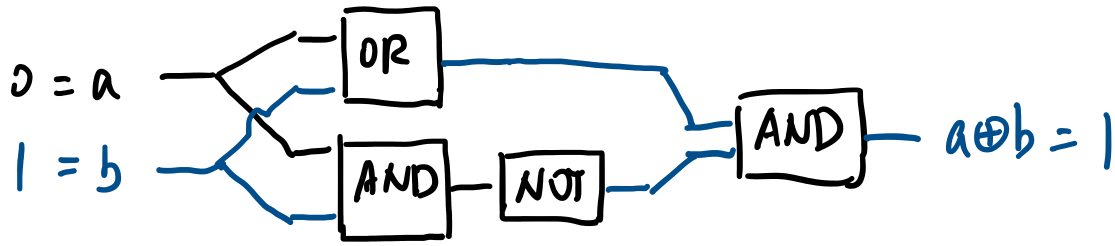
"parity"

Boolean circuit



$a \oplus b$

E.g. $a=0$, $b=1$



Reversible Computation.

Logical reversibility: A logic gate is reversible if the mapping from input to output is a bijection.

"Given the output, the input is uniquely determined."

Example: NOT gate is reversible.

Given output, we can always negate back to input.

Example: XOR function is <u>not</u> reversible.

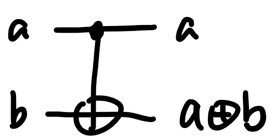
Given $a \oplus b = 1$, we cannot determine if we had:

$(a=0, b=1)$ or $(a=1, b=0)$.

Q: Is there a reversible version of <u>XOR</u>?

A: Introducing **controlled-Not** gate (**CNOT**)

Circuit diagram



(trick: save the input)

Truth table

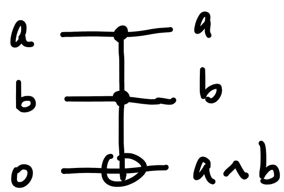
input	output
00	00
01	01
10	11
11	10

"Apply NOT to b if and only if $a=1$."

Bijection?

Q: Is the a reversible version of AND?

A: Introducing Toffoli gate.



ancilla

(trick: add ancilla input)

In general, third bit can be anything

output: $(a \wedge b) \oplus c$.

"Apply NOT to c if and only if both a and b are 1."

Q: Is there a reversible version for any Boolean function?

A: Theorem Any Boolean function $f: \{0,1\}^m \to \{0,1\}^n$ can be computed by a reversible circuit that consists of only Toffoli gates.*

k bits in, k out

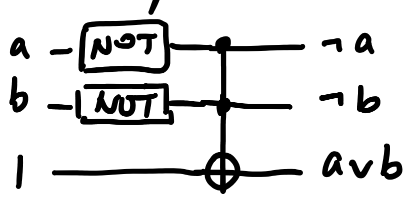
$k > m$, $k > n$.

"Reversibility + Universality".

(* assuming we allow extra ancilla inputs and some garbage outputs.)

How to show universality? (assume previous proposition)

Reversible OR:



de Morgan's Law

$a \vee b = \neg(\neg a \wedge \neg b)$

Reversible FANOUT:

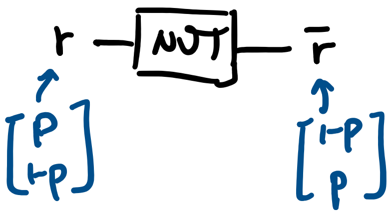
Later lecture: How many ancilla inputs needed?
How to clean up garbage bits?

Randomized Computation (random inputs, random gates)

A random bit r can be represented as a probablistic state vector

state of r : $\begin{bmatrix} p \\ 1-p \end{bmatrix}$ $\begin{matrix} \leftarrow \text{prob. of "0"} \\ \leftarrow \text{prob. of "1"} \end{matrix}$ $\quad 0 \le p \le 1$

What if we apply NOT gate to the random bit?

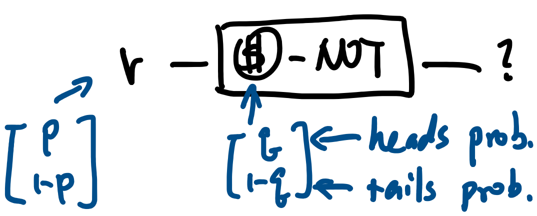


Matrix representation for this transition:

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{Pauli-}X}\begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} 1-p \\ p \end{bmatrix}.$$

What about a random gate?

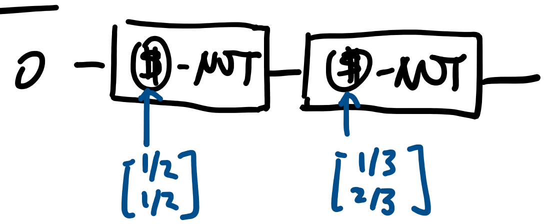
E.g. Randomized NOT gate: "Flips a coin; negate input if the outcome is tails."



$$\begin{bmatrix} ? \end{bmatrix}\begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} pq + (1-p)(1-q) \\ p(1-q) + q(1-p) \end{bmatrix}$$

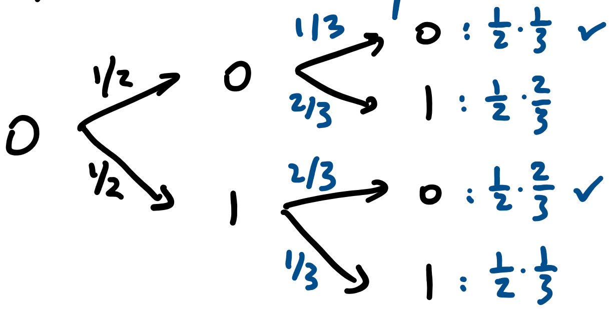
$$q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1-q)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q & 1-q \\ 1-q & q \end{bmatrix}$$

Example:



What is the probability that its output is 0?

This time, let's use computational path method:



$$\boxed{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{2}}$$

"Adding the probability of each correct path"

Quantum Computation

A quantum bit $|\psi\rangle$ can be represented as a superposition state vector:

Quantum State: $|\psi\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$ ← amplitude of $|0\rangle$, ← amplitude of $|1\rangle$. $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$

Let's see what happens if amplitudes are negative:

Introducing quantum gate: Hadamard gate (H)

Circuit Diagram

$-\boxed{H}-$

Truth Table

in	out
$\|0\rangle$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}} = \|+\rangle$
$\|1\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}} = \|-\rangle$
$\alpha\|0\rangle+\beta\|1\rangle$	$\alpha\|+\rangle+\beta\|-\rangle$

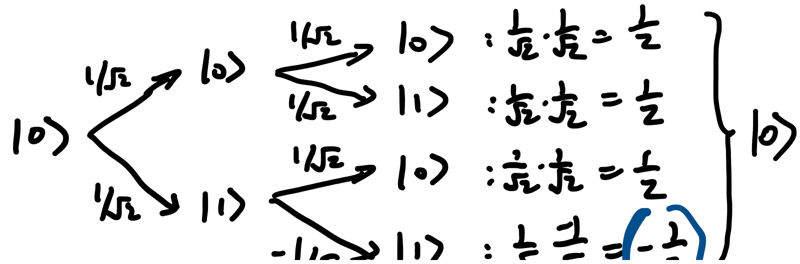
Matrix

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

Examples:

$\begin{bmatrix}0 \\ 1\end{bmatrix} = |1\rangle -\boxed{H}- ?$ $\quad \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} \\ -1/\sqrt{2}\end{bmatrix}$

$\begin{bmatrix}1 \\ 0\end{bmatrix} = |0\rangle -\boxed{H}-\boxed{H}- ?$



"Adding the amplitude of each path".

$|0\rangle : \frac{1}{2}+\frac{1}{2} = 1$: constructive interference

$|1\rangle : \frac{1}{2}-\frac{1}{2} = 0$: destructive interference.

Negative amplitudes can cancel with positive amplitudes.

This is another hint of the power of quantum computation.

- In fact, real (positive and negative) amplitudes are usually enough in most quantum algorithms.
- Complex amplitudes are need, however, to fully describe a quantum process. (See Aaronson's blog for sutlety)

Another (non-classical) gate: Square-root of NOT gate. ($\sqrt{X}$ gate).

What is a gate U, such that $U^2 = X$ gate?

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (+1)|+\rangle\langle +| + (-1)|-\rangle\langle -| \quad \text{(Spectral)}$$

$$\begin{aligned} X^{1/2} &= \sqrt{+1}\,|+\rangle\langle +| + \sqrt{-1}\,|-\rangle\langle -| \\ &= \frac{|0\rangle+|1\rangle}{\sqrt{2}} \cdot \frac{\langle 0|+\langle 1|}{\sqrt{2}} + i \cdot \frac{|0\rangle-|1\rangle}{\sqrt{2}} \cdot \frac{\langle 0|-\langle 1|}{\sqrt{2}} \\ &= \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}. \end{aligned}$$