Lecture 19. Basics of quantum error correction

Outline

- Quantum repetition code, logical qubit
- logical error rate v.s. physical error rate
- Key prerequisite of quantum error correction

As a student, I remember how confused I was when professor introduced various quantum codes and demonstrate how they can correct some certain types of errors. specifically Pauli errors.

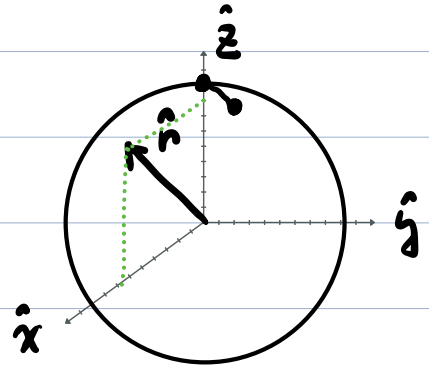
error $\hat{X}$, $\hat{Y}$, $\hat{Z}$

prob. P_x P_y P_z

"digital errors"

- $\hat{X}$: bit flip $\hat{X}|0\rangle = |1\rangle$, $\hat{X}|1\rangle = |0\rangle$
- $\hat{Z}$: phase flip $\hat{Z}|0\rangle = |0\rangle$, $\hat{Z}|1\rangle = -|1\rangle$
- $\hat{Y} = i\hat{X}\hat{Z}$ both

What about other types of errors? arbitrary unitary

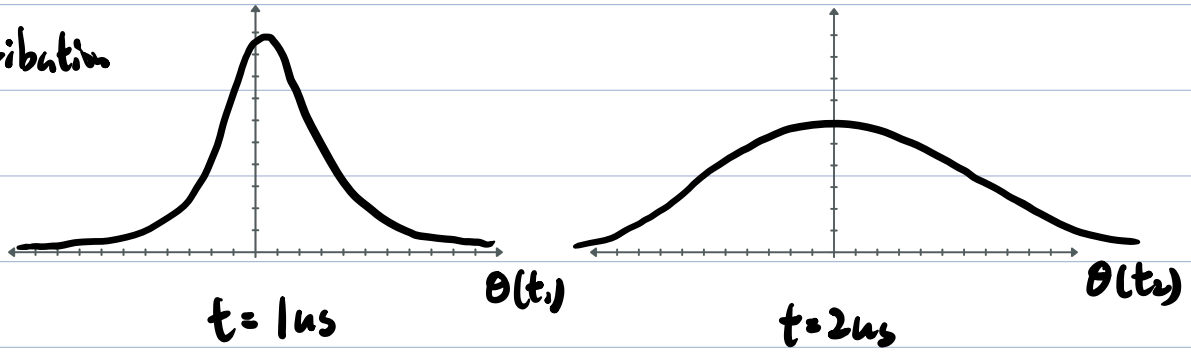


rotation axis $\hat{n} = r_x\hat{X} + r_y\hat{Y} + r_z\hat{Z}$

error: $\hat{R}_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}}$

"analog errors"

distribution



for a small t, θ is small

in classical circuit, ① digital errors are easy to correct.
because there are finite number of cases and one can round up to the nearest result or use classical codes to further correct them

② analog errors are hard to correct.
any value is a valid value, you cannot round it up to some "correct values"

Hopefully after this lecture, you're not confused like what I did. I'm going to change the game. I'll let you guys to propose what are the errors. Then we'll look at them together how to correct them.

Before starting, I want to remind you of what you've learned about in Monday's lecture.

Quantum channel: $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ Kraus operators

$\sum_k E_k^\dagger E_k = I$

ρ: density matrix.

for pure state $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$

① error $\hat{X}$, $\hat{Y}$, $\hat{Z}$ "digital errors"

prob. p_x p_y p_z

suppose $|\psi'\rangle = \hat{X}|\psi\rangle$, $\rho' = |\psi'\rangle\langle\psi'| = \hat{X}|\psi\rangle\langle\psi|\hat{X}^\dagger = \hat{X}\rho\hat{X}^\dagger$

$$\mathcal{E}(\rho) = (1-p_x-p_y-p_z)\, I\rho I^\dagger + p_x \hat{x}\rho\hat{x}^\dagger + p_y \hat{y}\rho\hat{y}^\dagger + p_z \hat{z}\rho\hat{z}^\dagger$$

$$= (1-p_x-p_y-p_z)\,\rho + p_x \hat{x}\rho\hat{x} + p_y \hat{y}\rho\hat{y} + p_z \hat{z}\rho\hat{z}$$

pure state $\xrightarrow{\mathcal{E}(\rho)}$ mixed state

(practice)

Kraus operators:

$$\begin{cases} \hat{E}_1 = \sqrt{1-p_x-p_y-p_z}\, \hat{I} \\ \hat{E}_2 = \sqrt{p_x}\, \hat{x} \\ \hat{E}_3 = \sqrt{p_y}\, \hat{y} \\ \hat{E}_4 = \sqrt{p_z}\, \hat{z} \end{cases}$$

(verify)

$$\sum_k \hat{E}_k^\dagger \hat{E}_k = (1-p_x-p_y-p_z)\,\hat{I} + p_x \hat{x}^\dagger\hat{x} + p_y \hat{y}^\dagger\hat{y} + p_z \hat{z}^\dagger\hat{z}$$

$$= I$$

② error: $\hat{R}_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}}$ rotation axis $\hat{n}$ "analog errors"

$$\mathcal{E}(\rho) = \hat{R}_{\hat{n}}(\theta)\,\rho\,\hat{R}_{\hat{n}}^\dagger(\theta) = e^{-i\frac{\theta}{2}\hat{n}}\,\rho\, e^{i\frac{\theta}{2}\hat{n}}$$

Kraus operator $\hat{E}_1 = \hat{R}_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}}$

pure state $\xrightarrow{\mathcal{E}(\rho)}$ pure state

$\hat{n} = \hat{x}, \hat{y}, \hat{z}$ or any unit length axis

if necessary, explain why $\hat{R}_{\hat{x}}(\pi) = \hat{X}$

③ what else?

non-unitary Kraus operator: e.g. "amplitude damping"

$|0\rangle \to |0\rangle$ $\quad$ $|1\rangle \to |0\rangle$

higher energy, spontaneous emission

with prob. of emission $= \gamma$

$$\mathcal{E}(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger$$

$$\begin{cases} K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \\ K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \end{cases}$$

(verify) ① $|\psi\rangle = |0\rangle$

$$\mathcal{E}(\rho) = K_0|\psi\rangle\langle\psi|K_0^\dagger + K_1|\psi\rangle\langle\psi|K_1^\dagger$$
$$= (K_0|\psi\rangle)(K_0|\psi\rangle)^\dagger + (K_1|\psi\rangle)(K_1|\psi\rangle)^\dagger$$
$$= |0\rangle\langle 0|$$

$|\psi'\rangle = |0\rangle$

② $|\psi\rangle = |1\rangle$

$$\mathcal{E}(\rho) = (K_0|\psi\rangle)(K_0|\psi\rangle)^\dagger + (K_1|\psi\rangle)(K_1|\psi\rangle)^\dagger$$
$$= (1-\gamma)|1\rangle\langle 1| + \gamma|0\rangle\langle 0|$$

$$\begin{cases} \text{with prob. } 1-\gamma : |\psi'\rangle = |1\rangle \\ \text{with prob. } \gamma : |\psi'\rangle = |0\rangle \end{cases}$$

(verify) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\begin{cases} K_0|\psi\rangle = \alpha|0\rangle + \beta\cdot\sqrt{1-\gamma}\,|1\rangle \\ K_1|\psi\rangle = \beta\sqrt{\gamma}\,|0\rangle \end{cases}$$

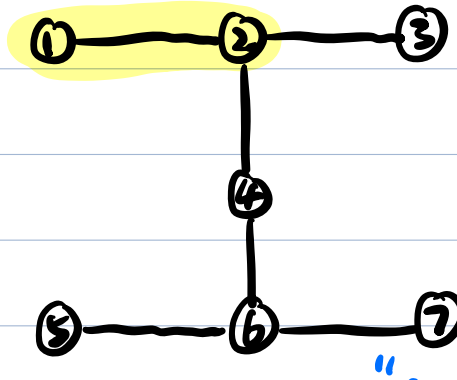
$$|K_1|4\rangle = \beta|0\rangle$$

$$\rho' = \mathcal{E}(\rho) = K_0|4\rangle\langle 4|K_0^\dagger + K_1|4\rangle\langle 4|K_1^\dagger$$
$$= (K_0|4\rangle)(K_0|4\rangle)^\dagger + (K_1|4\rangle)(K_1|4\rangle)^\dagger$$
$$= |4_1\rangle\langle 4_1| + \gamma \cdot \beta|0\rangle\langle 0|\beta^*$$

④ what else ?

cross-talk

IBM-Q Jakarta



"correlated digital error"

prob. p : $\hat{n}_1 \otimes \hat{n}_2$, $\mathcal{E}(\rho) = (1-p)\rho + (\hat{n}_1\hat{n}_2)\rho(\hat{n}_1^\dagger\hat{n}_2^\dagger)$

$\hat{R}_{\hat{n}_1\hat{n}_2}(\theta) = e^{-i\frac{\theta}{2}\hat{n}_1\hat{n}_2}$, $\mathcal{E}(\rho) = \hat{R}_{\hat{n}_1\hat{n}_2}(\theta)\,\rho\,\hat{R}^\dagger_{\hat{n}_1\hat{n}_2}(\theta)$

"correlated analog error"

Sort these error channels from easy to hard

Pauli / Single: $\hat{X}$

i) digital error : only bit-flip error channel

$$\mathcal{E}(\rho) = (1-p_x)\rho + p_x\hat{X}\rho\hat{X}$$

ii) analog error : small rotation along $\hat{X}$ axis

$$\mathcal{E}(\rho) = \hat{R}_{\hat{X}}(\theta)\,\rho\,\hat{R}^\dagger_{\hat{X}}(\theta) = e^{-i\frac{\theta}{2}\hat{X}}\rho\, e^{i\frac{\theta}{2}\hat{X}}$$

qubit

iii) digital error:

$$\mathcal{E}(\rho) = (1-p_x-p_y-p_z)\rho + p_x\hat{X}\rho\hat{X} + p_y\hat{Y}\rho\hat{Y} + p_z\hat{Z}\rho\hat{Z}$$

iv) analog error:

$$\mathcal{E}(\rho) = \hat{R}_{\hat{n}}(\theta)\,\rho\,\hat{R}_{\hat{n}}^{\dagger}(\theta) = e^{-i\frac{\theta}{2}\hat{n}}\,\rho\,e^{i\frac{\theta}{2}\hat{n}}$$

v) amplitude damping:

$$\mathcal{E}(\rho) = K_0\rho K_0^{\dagger} + K_1\rho K_1^{\dagger}$$

$$\begin{cases} K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \\ K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \end{cases}$$

multiple qubit

vi) correlated digital/analog error:

$$\mathcal{E}(\rho) = (1-p)\rho + p(\hat{n}_1\hat{n}_2)\rho(\hat{n}_1^{\dagger}\hat{n}_2^{\dagger})$$

$$\mathcal{E}(\rho) = \hat{R}_{\hat{n}_1\hat{n}_2}(\theta)\,\rho\,\hat{R}_{\hat{n}_1\hat{n}_2}^{\dagger}(\theta)$$

Quantum repetition code & logical qubit

solve i) digital error: only bit-flip error channel

$$\mathcal{E}(\rho) = (1-p_x)\rho + p_x\hat{X}\rho\hat{X}$$

Classical repetition code

bit-flip error channel $\begin{cases} \text{prob. } 1-p: \text{ no bit flip} \\ \text{prob. } p: \text{ bit flip} \end{cases}$

n bits, each bit suffers from bit flip independently

e.g. $n = 3$

logical 0 : 0 0 0

logical 1 : 1 1 1

↑ ↑ ↑ physical

storing logical 0

probability	state	corrected (majority vote)	
$(1-p)^3$	000	000	
$p(1-p)^2$	100	000	
$p(1-p)^2$	010	000	
$p(1-p)^2$	001	000	
$p^2(1-p)$	110	111	logical error
$p^2(1-p)$	101	111	prob. $= 3p^2(1-p) + p^3$
$p^2(1-p)$	011	111	
p^3	111	111	

logical error rate $= 3p^2(1-p) + p^3$

physical error rate $= p$

for small $p \ll 1$, $p^2 \ll p$

Quantum repetition code $n = 3$ qubits

logical $|0\rangle_L = |000\rangle$

logical $|1\rangle_L = |111\rangle$

but, we cannot measure individual qubits because

that would collapse the logical state

e.g. $|+\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L) = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$

once measured, it will collapse to either $|0\rangle_L = |000\rangle$ or $|1\rangle_L = |111\rangle$

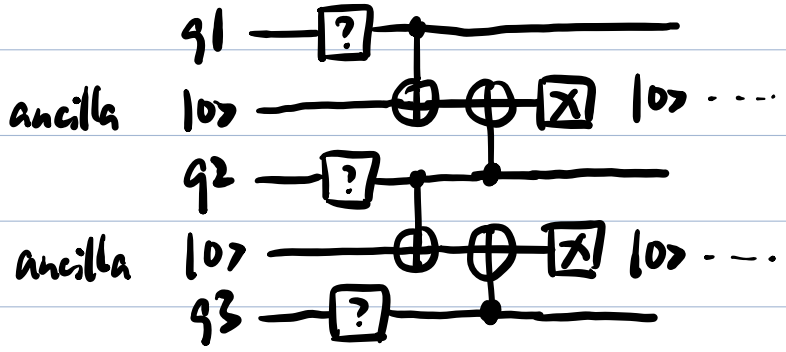
- How to detect bit flips?

indirectly measure the qubits.

$|0000\rangle$ or $|1111\rangle$

$\hat{O}_1 = \hat{Z}_1\hat{Z}_2\hat{I}_3$ measures $+1$

$\hat{O}_2 = \hat{I}_1\hat{Z}_2\hat{Z}_3$ measures $+1$



- How to correct bit flips?

i) digital error: only bit-flip error channel

$$\mathcal{E}(\rho) = (1-p_x)\rho + p_x\hat{X}\rho\hat{X}$$

e.g. $|0\rangle_L = |000\rangle$

probability	state	measurement	correction	corrected
$(1-p)^3$	$\|000\rangle$	$+1,+1$	$\hat{I}$	$\|000\rangle$
$p(1-p)^2$	$\|100\rangle$	$-1,+1$	$\hat{X}_1$	$\|000\rangle$
$p(1-p)^2$	$\|010\rangle$	$-1,-1$	$\hat{X}_2$	$\|000\rangle$
$p(1-p)^2$	$\|001\rangle$	$+1,-1$	$\hat{X}_3$	$\|000\rangle$
$p^2(1-p)$	$\|110\rangle$	$+1,-1$		logical $\begin{cases} \|111\rangle \\ \text{error} \end{cases}$

$p^2(1-p)$	$\|101\rangle$	$-1,-1$	logical error	$\|111\rangle$
$p^2(1-p)$	$\|011\rangle$	$-1,+1$		$\|111\rangle$
p^3	$\|111\rangle$	$+1,+1$		$\|111\rangle$

$$\begin{cases} \text{logical error rate} = 3p^2(1-p) + p^3 \\ \text{physical error rate} = p \end{cases} \quad \text{for small } p \ll 1,\ p^2 \ll p$$

generally n qubits (n is odd)

logical error rate $\sim p^{\frac{n+1}{2}}$, exponentially suppressed

---

ii) analog error: small rotation along $\hat{x}$ axis

$$\mathcal{E}(\rho) = \hat{R}_{\hat{x}}(\theta)\,\rho\,\hat{R}^{\dagger}_{\hat{x}}(\theta) = e^{-i\frac{\theta}{2}\hat{x}}\,\rho\,e^{i\frac{\theta}{2}\hat{x}}$$

still use quantum repetition code

$$|0\rangle_L = |000\rangle \qquad |1\rangle_L = |111\rangle$$

① suppose only error on the first qubit.

$$\hat{R}_{\hat{x}}(\theta) = e^{-i\frac{\theta}{2}\hat{x}} = \cos\frac{\theta}{2}\hat{I} - i\sin\frac{\theta}{2}\hat{X}$$

$$\left(\hat{R}_{\hat{x}}(\theta)\otimes\hat{I}_2\otimes\hat{I}_3\right)|000\rangle = \left(\cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle\right)\otimes|00\rangle$$

$$\hat{O}_1 = \hat{Z}_1\hat{Z}_2, \quad \hat{O}_2 = \hat{Z}_2\hat{Z}_3$$

$$\begin{cases} \text{prob. } \cos^2\frac{\theta}{2} & +1,+1 & |000\rangle \xrightarrow{\text{corrected}} |000\rangle \\ \text{prob. } \sin^2\frac{\theta}{2} & -1,+1 & |100\rangle \xrightarrow{\text{corrected}} |000\rangle \end{cases}$$

logical error rate = 0

② independent errors on all qubits

$$\left[\hat{R}_x(\alpha) \otimes \hat{R}_x(\beta) \otimes \hat{R}_x(\gamma)\right] |000\rangle$$

$$= \left(\cos\tfrac{\alpha}{2}|0\rangle - i\sin\tfrac{\alpha}{2}|1\rangle\right) \otimes \left(\cos\tfrac{\beta}{2}|0\rangle - i\sin\tfrac{\beta}{2}|1\rangle\right) \otimes \left(\cos\tfrac{\gamma}{2}|0\rangle - i\sin\tfrac{\gamma}{2}|1\rangle\right)$$

prob. $\left(\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\gamma}{2}\right)^2 + \left(\sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\gamma}{2}\right)^2$ +1, +1

$$\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\gamma}{2}|000\rangle + i\sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\gamma}{2}|111\rangle \quad \text{(normalize)}$$

given $\alpha, \beta, \gamma \ll 1$ $\approx |000\rangle + i\frac{\alpha\beta\gamma}{8}|111\rangle = |0\rangle_L + \frac{\alpha\beta\gamma}{8}|1\rangle_L$

- logical rotation : $\frac{\alpha\beta\gamma}{8}$ $\ll$ physical rotation
- physical rotation : $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$

prob. $\left(\sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}\cos\tfrac{\gamma}{2}\right)^2 + \left(\cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\sin\tfrac{\gamma}{2}\right)^2$ −1, +1

$$\approx -i\tfrac{\alpha}{2}|100\rangle - \tfrac{\beta\gamma}{4}|011\rangle \xrightarrow{\text{corrected}} |000\rangle - i\tfrac{\beta\gamma}{2\alpha}|111\rangle$$

- logical rotation : $\frac{\beta\gamma}{2\alpha}$ $\approx$ physical rotation
- physical rotation : $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$

failure ?

not necessarily, because prob. $\approx \left(\frac{\alpha}{2}\right)^2$

⋮

challenge: what if logical rotation is not tolerable ?

let's increase $n = 5$, physical rotation $= \frac{\theta}{2} \ll 1$

$$\begin{cases} \text{with high prob.} & +1,+1,+1,+1 & \approx |0\rangle_L + \left(\frac{\theta}{2}\right)^5 |1\rangle_L \\ \text{with prob.} \approx \left(\frac{\theta}{2}\right)^2 & -1,+1,+1,+1 \\ & +1,-1,+1,+1 & \approx |0\rangle_L + \left(\frac{\theta}{2}\right)^3 |1\rangle_L \\ \vdots & \vdots & \\ \text{with prob.} \propto \left(\frac{\theta}{2}\right)^4 & & \approx |0\rangle_L + \left(\frac{\theta}{2}\right) |1\rangle_L \\ \vdots & & \uparrow \text{logical rotation} \end{cases}$$

for arbitrary n (n is odd)

$$\begin{cases} \text{with high prob.} & \approx |0\rangle_L + \left(\frac{\theta}{2}\right)^n |1\rangle_L \\ \text{with prob.} \approx \left(\frac{\theta}{2}\right)^2 & \approx |0\rangle_L + \left(\frac{\theta}{2}\right)^{n-2} |1\rangle_L \\ \approx \left(\frac{\theta}{2}\right)^4 & \propto |0\rangle_L + \left(\frac{\theta}{2}\right)^{n-4} |1\rangle_L \\ \vdots & \uparrow \text{logical rotation} \end{cases}$$

logical rotation is exponentially suppressed.

if you can tolerate only $\left(\frac{\theta}{2}\right)^k$ prob. of $\left(\frac{\theta}{2}\right)^l$ logical rotation

then $n \geq k + l$

---

iii) digital error:

$$\mathcal{E}(\rho) = (1 - p_x - p_y - p_z)\rho + p_x \hat{x}\rho\hat{x} + p_y \hat{y}\rho\hat{y} + p_z \hat{z}\rho\hat{z}$$

quantum repetition code to detect phase flip $\hat{z}$ error

$$|0\rangle_L = |+++\rangle \qquad |1\rangle_L = |---\rangle$$

measure $\hat{O}_1 = \hat{X}_1\hat{X}_2$ $\hat{O}_2 = \hat{X}_2\hat{X}_3$

e.g. $|0\rangle_L = |+++\rangle$

probability	state	measurement	correction	corrected
$(1-p)^3$	$\|+++\rangle$	+1, +1	$\hat{I}$	$\|+++\rangle$
$p(1-p)^2$	$\|-++\rangle$	-1, +1	$\hat{Z}_1$	$\|+++\rangle$
$p(1-p)^2$	$\|+-+\rangle$	-1, -1	$\hat{Z}_2$	$\|+++\rangle$
$p(1-p)^2$	$\|++-\rangle$	+1, -1	$\hat{Z}_3$	$\|+++\rangle$
$p^2(1-p)$	$\|--+\rangle$	+1, -1	logical error	$\|--\rangle$
$p^2(1-p)$	$\|-+-\rangle$	-1, -1		$\|---\rangle$
$p^2(1-p)$	$\|+--\rangle$	-1, +1		$\|---\rangle$
p^3	$\|---\rangle$	+1, +1		$\|---\rangle$

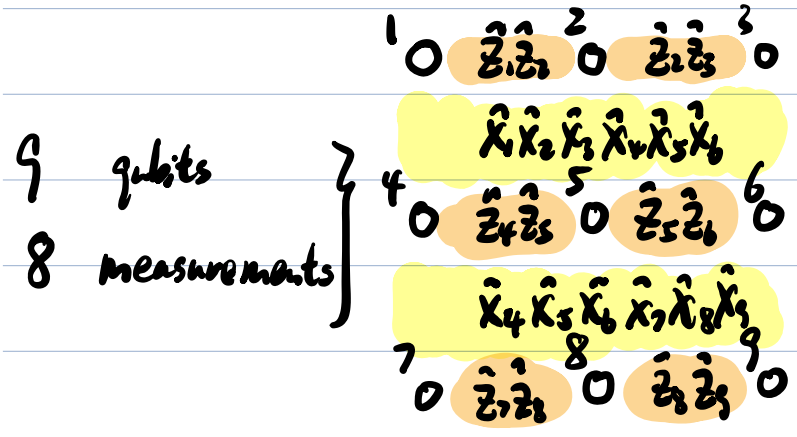
Concatenated quantum repetition code

$|0\rangle_L = |+++\rangle$

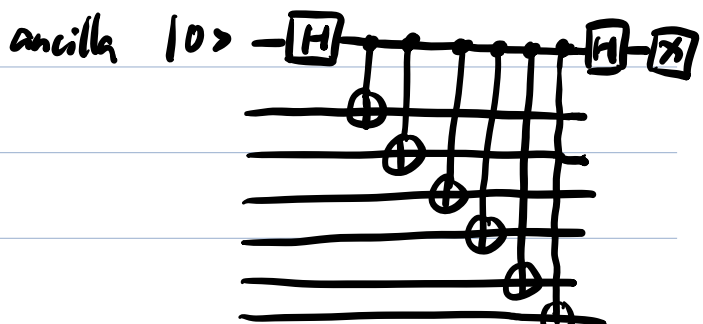
↑ ↑ ↑

each qubit is logical qubit. $|+\rangle_L = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$

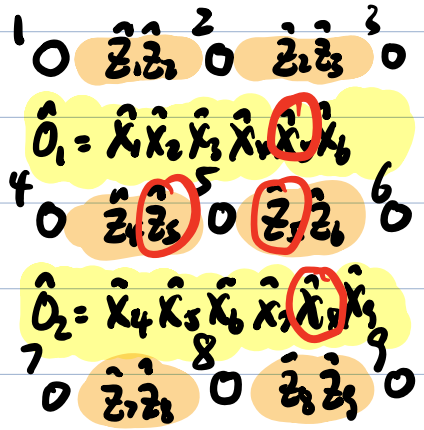
9 qubit can correct any single Pauli error



$\hat{X}_1\hat{X}_2\hat{X}_3\hat{X}_4\hat{X}_5\hat{X}_6$



practice:



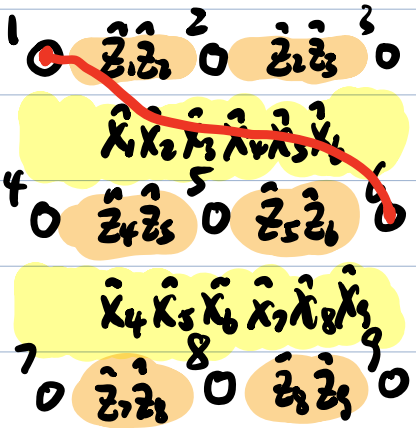
$\hat{X}_5$ $\hat{Z}_4\hat{Z}_5$, $\hat{Z}_5\hat{Z}_6$: -1

$\hat{Z}_5$ $\hat{O}_1, \hat{O}_2$: -1

$\hat{Y}_5$ $\hat{Z}_4\hat{Z}_5$, $\hat{Z}_5\hat{Z}_6$, O_1

$$\begin{cases} |0\rangle_L = \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \\ |1\rangle_L = \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3} \end{cases}$$

- Theoretically, any logical qubit is a normal qubit, so it can be used as "physical qubit": concatenation
- In practice, it may require long-distance gates (transversal gates) whose error rate may be proportional to distance



longest gate distance $\propto \sqrt{n}$

Next few lectures: Topological quantum codes

nearest-neighbor interaction

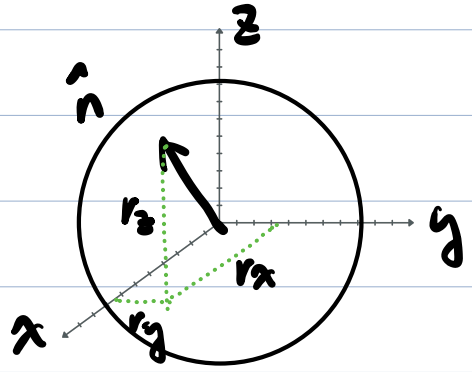
locality $\Rightarrow$ constant physical error rates

iv) analog error:

$$\mathcal{E}(\rho) = \hat{R}_{\hat{n}}(\theta)\,\rho\,\hat{R}_{\hat{n}}^{\dagger}(\theta) = e^{-i\frac{\theta}{2}\hat{n}}\,\rho\,e^{i\frac{\theta}{2}\hat{n}}$$

$$e^{-i\frac{\theta}{2}\hat{n}} = \cos\frac{\theta}{2}\cdot I - i\sin\frac{\theta}{2}\,\hat{n} \qquad \hat{n}: \text{rotation axis}$$

$$= \cos\frac{\theta}{2}\cdot I - i\sin\frac{\theta}{2}\left(r_x\hat{X} + r_y\hat{Y} + r_z\hat{Z}\right)$$



As long as a code can correct Pauli error, it can correct the corresponding analog error as well

$$\hat{X},\ p_x \ll 1 \longrightarrow \hat{R}_{\hat{x}}(\theta),\ \theta \ll 1$$

$$\hat{X},\hat{Y},\hat{Z}\quad p_x, p_y, p_z \ll 1 \longrightarrow \hat{R}_{\hat{n}}(\theta),\ \theta \ll 1,\ \text{any } \hat{n}$$

v) amplitude damping:

$$\mathcal{E}(\rho) = K_0\rho K_0^{\dagger} + K_1\rho K_1^{\dagger}$$

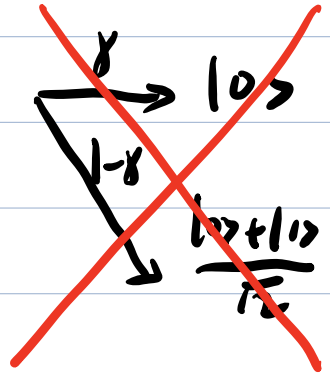
$$\begin{cases} K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \\ K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \end{cases}$$

$$|0\rangle \longrightarrow |0\rangle$$

$$|1\rangle \xrightarrow{\gamma} |0\rangle$$

$$|1\rangle \xrightarrow{1-\gamma} |1\rangle$$

$|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \longrightarrow ?$



$$\begin{cases} K_0|+\rangle = \dfrac{|0\rangle+\sqrt{1-\gamma}\,|1\rangle}{\sqrt{2}} \\ K_1|+\rangle = \dfrac{\sqrt{\gamma}\,|0\rangle}{\sqrt{2}} \end{cases}$$

$$\mathcal{E}(\rho) = K_0|+\rangle\langle+|K_0^\dagger + K_1|+\rangle\langle+|K_1^\dagger$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{1-\gamma} \\ \sqrt{1-\gamma} & 1-\gamma \end{pmatrix} + \frac{\gamma}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\gamma & \sqrt{1-\gamma} \\ \sqrt{1-\gamma} & 1-\gamma \end{pmatrix} \neq (1-\gamma)|+\rangle\langle+| + \gamma|0\rangle\langle 0|$$

$$\frac{1}{2}\begin{pmatrix} 1+\gamma & 1-\gamma \\ 1-\gamma & 1-\gamma \end{pmatrix}$$

"digital error" + "analog error"

practice yourself:

$$\begin{cases} \text{quantum repetition code cannot correct this } |+\rangle_L \\ \text{concatected quantum repetition code can correct this} \end{cases}$$

---

vi) correlated digital/analog error:

$$\mathcal{E}(\rho) = (1-p)\rho + p\,(\hat{n}_1\hat{n}_2)\,\rho\,(\hat{n}_1^\dagger \hat{n}_2^\dagger)$$

$$\mathcal{E}(\rho) = \hat{R}_{\hat{n}_1\hat{n}_2}(\theta)\;\rho\;\hat{R}^\dagger_{\hat{n}_1\hat{n}_2}(\theta)$$

e.g. $\mathcal{E}(\rho) = (1-p)\rho + (\hat{X}_1\hat{X}_2)\,\rho\,(\hat{X}_1^\dagger \hat{X}_2^\dagger)$

prob p $\hat{X}_1\hat{X}_2$, $\hat{X}_2\hat{X}_3$, $\hat{X}_1\hat{X}_3$

probability	state	measurement	correction	corrected
$(1-p)^3$	$\|000\rangle$	$+1,+1$	$\hat{I}$	$\|000\rangle$
$p^2(1-p)$	$\|100\rangle$	$-1,+1$	$\hat{X}_1$	$\|000\rangle$
$p^2(1-p)$	$\|010\rangle$	$-1,-1$	$\hat{X}_2$	$\|000\rangle$
$p^2(1-p)$	$\|001\rangle$	$+1,-1$	$\hat{X}_3$	$\|000\rangle$
$p(1-p)^2$	$\|110\rangle$	$+1,-1$	logical error	$\|111\rangle$
$p(1-p)^2$	$\|101\rangle$	$-1,-1$		$\|111\rangle$
$p(1-p)^2$	$\|011\rangle$	$-1,+1$		$\|111\rangle$
p^3	$\|111\rangle$	$+1,+1$		$\|111\rangle$

more likely.

a single correlated error causes logical error in $n=3$ repetition code

in $n=5$ repetition code, two correlated error causes logical error

for n repetition code.

$$\begin{cases} \frac{n+1}{F} & \text{correlated error causes logical error.} \\ \frac{n+1}{2} & \text{single error causes logical error} \end{cases}$$

still exponentially suppressing "local" correlated error

what if errors are not local?

any n: $\bigotimes_{i=1}^{n} \hat{X}_i$ with prob. p

logical error rate $\sim p$, cannot suppress

Key prerequisite of quantum error correctio

* errors are "local"

generally, single error must be othogonal to logical operator.
measurements will project physical erros and correct based

both "digital errors" and "analog errors" } correctable

$\hat{X}, \hat{Y}, \hat{Z}$ $\quad$ $\hat{R}_{\hat{n}}(\theta)$

other types of non-unitary errors