

Lecture 2 CPSC 447/547 - Intro to QC

Mathematics Toolkit (for quantum computation)

Outline (Basic linear algebra)

- (Complex) vector spaces .
 - Linear operators : projections .
 - Spectral Theorem
-

Linear algebra is the language
for describing (the dynamics of) quantum systems.

From last time, the state of a quantum bit:

"superposition of 0 and 1": $\underline{\alpha}|0\rangle + \underline{\beta}|1\rangle$
linear combination "amplitudes"

The state of a qubit can be represented as :

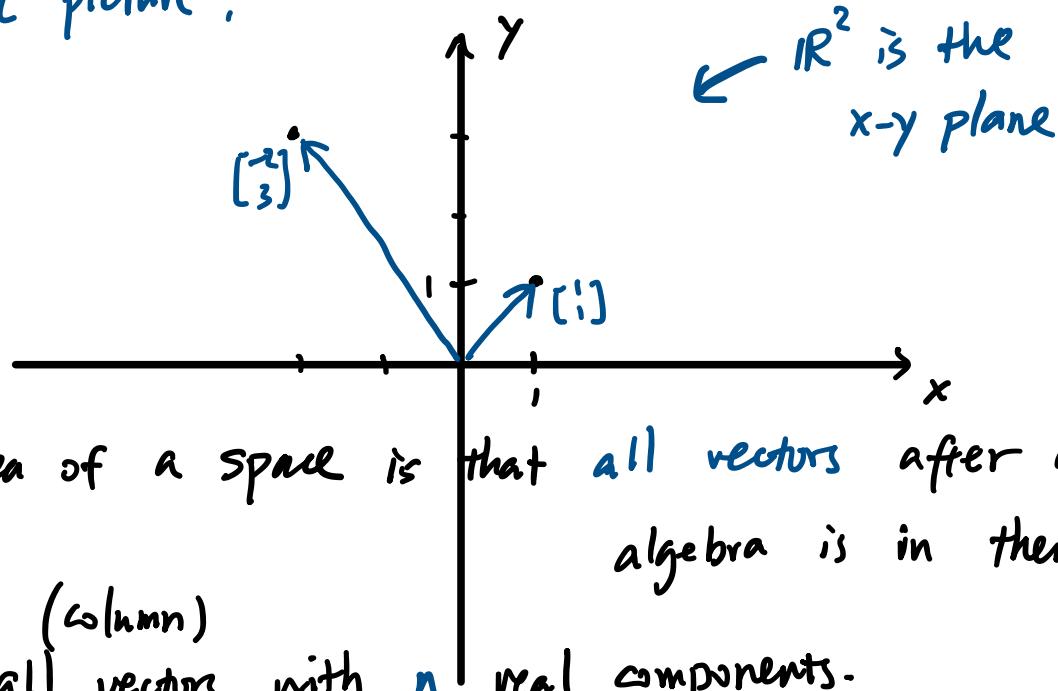
a unit (column) vector in \mathbb{C}^2 plane.

Let's start with a simpler example: vector space \mathbb{R}^2 .

- Vector space : a set of vectors that allows us "space" to take linear combinations.
- Algebra : vector addition , scalar multiplication

- \mathbb{R}^2 : 2-dim real $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

- Geometric picture:



The idea of a space is that all vectors after doing algebra is in there

\mathbb{R}^n : all ^(column) vectors with n real components.

More things we can say about vectors?

- Length of a vector: $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$,

$$\|v\|^2 = v_1^2 + v_2^2 = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ row multiplies col.}$$

$$= v^T v \text{ "transpose".}$$

- Angle between two vectors, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

 "Inner product" $w^T v = w_1 v_1 + w_2 v_2 = \|w\| \cdot \|v\| \cos \theta$

If $w^T v = 0$, then orthogonal.

Now, do they work if we have complex vectors?

Example : \mathbb{C}^2 , 2-dim complex vector space

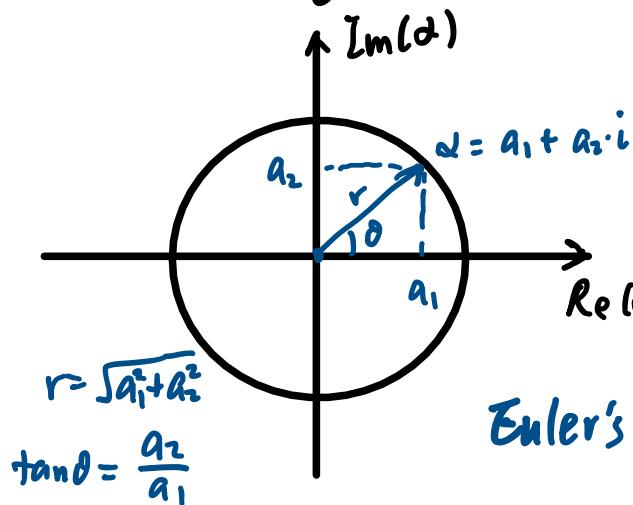
$$|v\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \xrightarrow{\text{complex numbers}} \text{"ket"}$$

- A complex number $\alpha \in \mathbb{C}$ is of form :

$\alpha = \underbrace{a_1}_{\text{real component}} + \underbrace{a_2 i}_{\text{imaginary component}}$

where $a_1, a_2 \in \mathbb{R}$ and $i^2 = -1$.

- We have a geometric picture (a complex # = 2 real #'s)



Notation : α^* is called the complex conjugate of α .

$$\alpha = a_1 + a_2 i$$

$$\alpha^* = a_1 - a_2 i \leftarrow \text{where on complex plane?}$$

Back to complex vectors : $|v\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}$

- Length : $(|v\rangle)^T |v\rangle$? $[1 \ i] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + i^2 = 0 \quad (\text{:(}) \quad \alpha^* \alpha = |\alpha|^2$

$$(|v\rangle^*)^T |v\rangle = [\alpha^* \ \beta^*] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha^* \alpha + \beta^* \beta = |\alpha|^2 + |\beta|^2 \quad (\text{:})$$

$$\langle v | = (\lvert v \rangle)^* = (\lvert v \rangle^*)^T$$

"bra"

adjoint / Hermitian : conjugate transpose

- Inner product (for complex vectors)

$$\langle w | \cdot | v \rangle \underset{j}{\equiv} \langle w | v \rangle = [w_1^* \ w_2^*] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = w_1^* v_1 + w_2^* v_2$$

matrix multiplication

→ Higher dimension \mathbb{C}^n :

$$\langle w | v \rangle = \sum_{j=1}^n w_j^* v_j$$

While we are at it, let's see what else needs

to be changed for complex.

$$\text{"Symmetry"} \quad \underline{\text{Symmetric matrix}} \quad \Rightarrow \quad \underline{\text{Hermitian matrix}}$$

$$A = A^T \qquad \qquad \qquad A = A^*$$

Orthogonal vectors v_1, \dots, v_n (of unit length) "orthonormal"

We used to have Now with complex components

$$v_j^T \cdot v_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} \Rightarrow \quad \langle v_j | v_k \rangle = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$$\text{"Orthogonality"} \quad \underline{\text{Orthogonal matrix}} \quad \underline{\text{Unitary matrix}}$$

(square)

(rows and cols of orthogonal vectors) \Rightarrow

$$Q = \begin{pmatrix} | & | \\ v_1 & \dots & v_n \\ | & | \end{pmatrix}$$

$$Q^T Q = I$$

$$U = \begin{pmatrix} | & | \\ |\lvert v_1 \rangle & \dots & |\lvert v_n \rangle \\ | & | \end{pmatrix}$$

$$U^T U = I$$

$$(Q^T = Q^{-1})$$

$$(U^+ = U^{-1})$$

Linear independence and basis

Vector space \mathbb{C}^2 , $\begin{bmatrix} 1 \\ i \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1+i \end{bmatrix}$, $\begin{bmatrix} 1-i \\ 2-3i \end{bmatrix}$

- A set of vectors $|v_1\rangle, |v_2\rangle, \dots$ **Spans** the vector space S
means: any vector in S can be written as
a linear combination of those vectors.
(with complex coefficients)

$$|w\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \dots + \alpha_n|v_n\rangle, \alpha_1, \dots, \alpha_n \in \mathbb{C}$$

- A set of (non-zero) vectors are **linearly independent**
means: no linear combination gives zero vector,
unless all coefficients are zero.

$$\alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \dots + \alpha_n|v_n\rangle \neq 0$$

(If some non-zero combination does give zero, then dependent)

Example: $|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- When the set of vectors $\left\{ \begin{array}{l} \text{spans the vector space } S \\ \text{and} \\ \text{they are linearly independent} \end{array} \right.$

we call them a **basis** for S .

And the number of vectors in the basis set

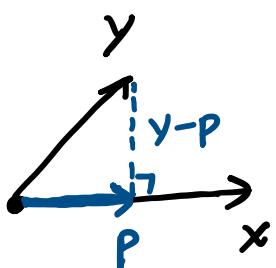
is called the dimension of S .

Projections (an example of linear operator)

What is a projection?

maps one vector space
to another

Example:



Projection of $|y\rangle$ onto the line of $|x\rangle$

want to find

$$|p\rangle = p|x\rangle.$$

" $y-p$ is perpendicular to x ": $\langle x| \cdot (|y\rangle - |p\rangle) = 0$

$$\langle x|y\rangle - p\langle x|x\rangle = 0$$

$$\Rightarrow p = \frac{\langle x|y\rangle}{\langle x|x\rangle}, \quad |p\rangle = \frac{|x\rangle \langle x|y\rangle}{\langle x|x\rangle}$$

Projection is a matrix

$$|p\rangle = \Pi \cdot |y\rangle, \quad \Pi = \frac{|x\rangle \langle x|}{\langle x|x\rangle} \leftarrow \begin{array}{l} \text{outer product} \\ (\text{matrix}) \end{array}$$

$$\leftarrow \begin{array}{l} \text{inner product} \\ (\text{scalar}) \end{array}$$

"Outer product of two vectors":

$$|w \times v| = |w\rangle (|v\rangle)^+ = \begin{pmatrix} w_1 v_1^* & w_1 v_2^* \\ w_2 v_1^* & w_2 v_2^* \end{pmatrix}$$

(e.g. \mathbb{C}^2)

Projection matrix Π takes $|y\rangle$ to the projected vector $|p\rangle$.

Properties: ① Π is symmetric (in the adjoint sense)

$$\Pi^+ = \Pi$$

② $\Pi^2 = \Pi$

- projection twice :

$$\Pi^2 = \Pi.$$

Eigenvectors and Eigenvalues

- **Eigenvalue equation** for a linear operator A :

$$A|j\rangle = \lambda_j |j\rangle .$$

where $|j\rangle$ is the (non-zero) eigenvector, and

λ_j is a complex number known as the eigenvalue.

Example : σ_z is a linear operator for vector space \mathbb{C}^2 :

"Pauli Z operator" $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Calculate eigenvectors and eigenvalues:

$$\det(\sigma_z - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = -1 .$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = -\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$= |0\rangle \quad = |1\rangle .$$

- **Spectral Theorem** for linear operators:

A linear operator is **normal** if $A^*A = AA^*$

Thm. A normal operator can be represented in its spectral decomposition:

$$A = \sum_j \lambda_j |j\rangle X_j | .$$

where the vectors $|j\rangle$ are orthonormal eigenvectors of A and λ_j are their corresponding eigenvalues.

Example : $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (+) \underbrace{|0\rangle X_0|}_{\text{projector}} + (-) \underbrace{|1\rangle X_1|}_{\text{projector}}$

projectors corresponding to eigenvectors.

Example : $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = ?$

$$= (+) |+X+| + (-) |-X-| .$$

$$\text{where } |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) , |- \rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) .$$

Application of Spectral Theorem

Power of matrix : why? orthonormal :

$$A^3 = \left(\sum_j \lambda_j |j\rangle X_j | \right)^3 = \sum_j \lambda_j^3 |j\rangle X_j |$$

Matrix exponential :

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \quad \text{power series}$$

$$= \sum_j e^{\lambda_j} |j\rangle X_j | .$$

Aside:

Outer product

$$r_u \gamma [w_i^* \ w_k^*]$$

$$|v\rangle \cdot \langle w| \equiv |v \times w| = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$$

$$= \begin{bmatrix} v_1 w_1^* & v_1 w_2^* \\ v_2 w_1^* & v_2 w_2^* \end{bmatrix}$$

Tensor product

$$|v\rangle \otimes |w\rangle \equiv |v, w\rangle \equiv |v\rangle |w\rangle$$

$$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{bmatrix}$$