

Lecture 4 CPSC 447/547 - Intro to QC

Superposition and Entanglement.

Outline

- Single-qubit state
- Multi-qubit state.

Four Principles of QC

- Superposition
 - Composition
 - Transformation
 - Measurement
- } quantum states : Today
} quantum gates

Superposition Principle

The state of a quantum bit (i.e., qubit) is a vector in a 2-dim complex Hilbert space \mathcal{H} .

In Dirac's notation : \uparrow vector space with distance defined by inner product.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \leftarrow \text{state vector}$$

- $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are computational **basis** vectors of \mathcal{H} .

- Inner Product of $|\Psi_1\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$, $|\Psi_2\rangle = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$:

$$\langle \Psi_1 | \Psi_2 \rangle = [\alpha_1^* \ \beta_1^+] \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \alpha_1^* \alpha_2 + \beta_1^* \beta_2$$

- Normalization :

$$\| |\Psi\rangle \| = \sqrt{\langle \Psi | \Psi \rangle} = 1 \quad (\text{i.e., } |\alpha|^2 + |\beta|^2 = 1)$$

Examples: $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$

$$-|0\rangle, i|1\rangle, \sqrt{\frac{2}{3}}|0\rangle + e^{i\pi/4}\sqrt{\frac{1}{3}}|1\rangle, \text{ etc.}$$

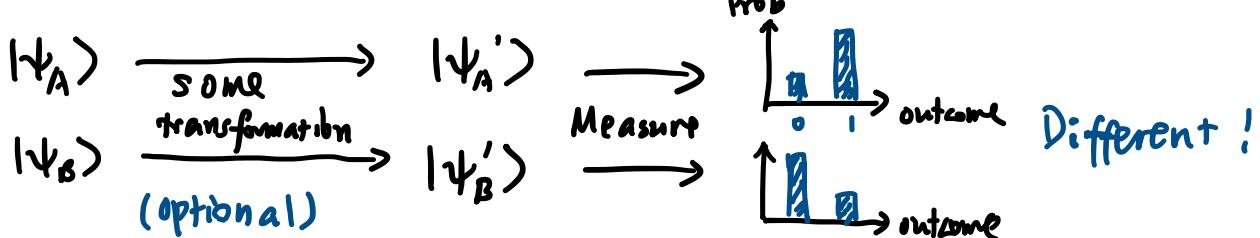
Unlike a probabilistic state vector, amplitudes can be negative or complex. From previous lecture, we saw how negative amplitudes can lead to interference.

Let's investigate this notion of complex superposition further.

Distinguishing two qubits (via measurements)

Given two qubits $|\Psi_A\rangle$, $|\Psi_B\rangle$, how do we

tell if they are different (through physical experiment)?



Example.

$$\left\{ |\Psi_A\rangle = |0\rangle \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ look "0"} \right.$$

$$|\Psi_B\rangle = |1\rangle \xrightarrow{\text{Measure}} \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \text{0\% "0"} \\ \text{0\% "1"} \end{array} \quad \text{Different.}$$

$$\begin{cases} |\Psi_A\rangle = |0\rangle \\ |\Psi_B\rangle = |+\rangle \end{cases} \xrightarrow{\text{Measure}} \begin{array}{c} \uparrow \\ \uparrow \end{array} \quad \begin{array}{c} 100\% "0" \\ 50\% "0" \end{array} \quad \text{Different.}$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\begin{cases} |\Psi_A\rangle = |+\rangle \\ |\Psi_B\rangle = |- \rangle \end{cases} \xrightarrow{\text{Measure}} \begin{array}{c} \uparrow \text{[A]} \\ \uparrow \text{[B]} \end{array} \quad \begin{array}{c} 50\% "0" \\ 50\% "0" \end{array} \quad \text{Same?}$$

$$= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\begin{array}{ccc} |+\rangle & \xrightarrow{H} & |0\rangle & \xrightarrow{\text{Measure}} & \begin{array}{c} \uparrow \text{[A]} \\ \uparrow \text{[B]} \end{array} & \begin{array}{c} 100\%, "0" \\ 0\%, "0" \end{array} \\ |-\rangle & \xrightarrow{H} & |1\rangle & \xrightarrow{\text{Measure}} & \begin{array}{c} \uparrow \text{[B]} \\ \uparrow \text{[A]} \end{array} & \text{Different.} \end{array}$$

$$\begin{cases} |\Psi_A\rangle = |0\rangle \\ |\Psi_B\rangle = -|0\rangle \end{cases} \xrightarrow{\text{Measure}} \begin{array}{c} \uparrow \text{[A]} \\ \uparrow \text{[B]} \end{array} \quad \text{Same?}$$

Global Phase: No physical experiment can distinguish $|0\rangle$ from $-|0\rangle$.

Similarly for $|0\rangle$ and $i|0\rangle$, etc.

In fact, cannot distinguish $|\Psi\rangle$ from $e^{i\theta}|\Psi\rangle$
for any θ .

A more succinct representation of quantum state
is:

$$|\Psi\rangle = \begin{bmatrix} e^{i\theta_1} \cdot a_1 \\ e^{i\theta_2} \cdot a_2 \end{bmatrix} = e^{i\theta_1} \begin{bmatrix} a_1 \\ e^{i(\theta_2 - \theta_1)} \cdot a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ e^{i\phi} \cdot a_2 \end{bmatrix}$$

where $a_1, a_2 \geq 0$ $a_1^2 + a_2^2 = 1$ $\phi \in [0, 2\pi)$ 3 real's.

This is great, because we can embed a qubit in \mathbb{R}^3 and visualize!

Visualization of a qubit (Bloch Sphere)

Let's pick a good coordinate system in 3d:

- Cartesian: (x, y, z)
- Spherical: $(r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta)$

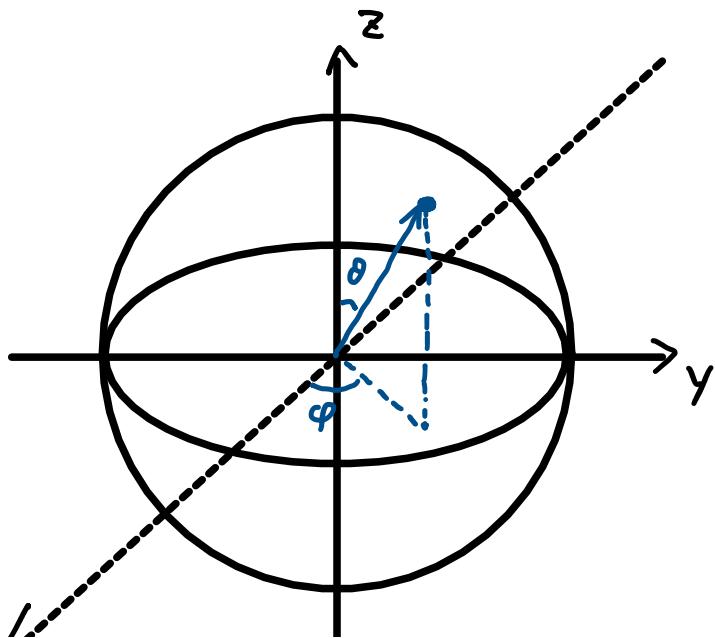
$$r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi$$

$$|\psi\rangle = \begin{bmatrix} a_1 \\ e^{i\varphi} a_2 \end{bmatrix}, \quad \|\langle\psi|\psi\rangle\|^2 = r^2 = a_1^2 + a_2^2 = 1.$$

\Downarrow Embedding on surface of a unit ball.

$$\begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi} \sin\frac{\theta}{2} \end{bmatrix} \quad \text{where } 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi$$

are angles in spherical coordinate.



\leftarrow
x

Example:

$$\textcircled{1} \quad (x, y, z) = (0, 0, 1) \Rightarrow (\theta, \varphi) = (0, 0)$$

anything
↓

$$\Rightarrow |\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle.$$

$$\textcircled{2} \quad (x, y, z) = (0, 0, -1) \Rightarrow (\theta, \varphi) = (\pi, 0)$$

$$\Rightarrow |\psi\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle.$$

Remark: Orthogonal states = opposite direction.

Let's try more examples on software (Mathematica).

What if we have more qubits?

Composition Principle

The joint state of two quantum systems, A and B, is the **tensor product** of $|\psi_A\rangle$ and $|\psi_B\rangle$:

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

$$= \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \\ \alpha_1 \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha_0 \beta_0 \\ \alpha_0 \beta_1 \\ \alpha_1 \beta_0 \\ \alpha_1 \beta_1 \end{bmatrix}$$

New Basis

$\alpha_0 \beta_0 \leftarrow |00\rangle = |0\rangle \otimes |0\rangle$
 $\alpha_0 \beta_1 \leftarrow |01\rangle$
 $\alpha_1 \beta_0 \leftarrow |10\rangle$
 $\alpha_1 \beta_1 \leftarrow |11\rangle$

Quick check: is $|\psi_{AB}\rangle$ a valid state?

$$|\alpha_0 \beta_0|^2 + |\alpha_0 \beta_1|^2 + |\alpha_1 \beta_0|^2 + |\alpha_1 \beta_1|^2$$

$$= (|\alpha_0|^2 + |\alpha_1|^2) \cdot (|\beta_0|^2 + |\beta_1|^2) = 1$$

Q: Given an arbitrary 2-qubit quantum state,
can we always write it as a tensor product?

Example ① $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle = \underbrace{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)}_{|\psi_A\rangle} \otimes \underbrace{|0\rangle}_{|\psi_B\rangle}$.

② $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$

The states that cannot be expressed as a tensor product of two (sub)states are called **entangled states**.

Those that can be expressed are **product states**.

In fact, most states are entangled.

Now, let's remember this special state:

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) : \text{"Bell state", "EPR pair", ...}$$

In principle, the two qubits of $|\psi_{AB}\rangle$ can be far away!
Why does this seem troubling?

"**EPR Paradox**" / "**Spooky action in a distance**"
Einstein - Podolsky - Rosen (1935).

Suppose Alice and Bob each takes one of the qubits of the Bell state. Then Alice flies to Mars with the qubit in her pocket.

What if Alice measures her qubit? (More on partial measurement later.)

$$|\Psi_{AB}\rangle = \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

Probability of getting

$$\left\{ \begin{array}{l} |00\rangle : |\alpha_{00}|^2 \\ |01\rangle : |\alpha_{01}|^2 \\ |10\rangle : |\alpha_{10}|^2 \\ |11\rangle : |\alpha_{11}|^2 \end{array} \right.$$

Probability of $|0\rangle$ for Alice : $|\alpha_{00}|^2 + |\alpha_{01}|^2$
(first qubit)

because outcome $|00\rangle$ and $|01\rangle$ are compatible with $|0\rangle$ for Alice.

Probability of $|1\rangle$ for Alice : $|\alpha_{10}|^2 + |\alpha_{11}|^2$.

Example EPR pair $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

Alice measures her qubit. she gets $|0\rangle$ w.p. $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

If Alice sees $|0\rangle$, she immediately knows that

- Bob will see $|0\rangle$ if he measures, because that's the only compatible outcome with Alice's measurement.
- In other words, **Bob's state collapses to $|0\rangle$** , if Alice sees $|0\rangle$ from her measurement.

Things get weird if Alice applies H gate then measures.

$$\frac{1}{\sqrt{2}}|00\rangle \xrightarrow{\text{H}_A} \frac{1}{\sqrt{2}}|00\rangle : \frac{1}{2} \quad \xrightarrow{\text{If Alice sees } |0\rangle :} \frac{1}{2}|0\rangle \otimes (\underline{|0\rangle + |1\rangle})$$

$$\begin{array}{c} \langle \quad | \quad \rangle \\ \frac{1}{\sqrt{2}} |11\rangle \end{array} \xrightarrow{\text{If Alice sees } |1\rangle:} \begin{array}{l} \cancel{\frac{1}{\sqrt{2}} |10\rangle : \frac{1}{2}} \cdot \cancel{\text{If Alice sees } |1\rangle:} \\ \cancel{-\frac{1}{\sqrt{2}} |11\rangle : -\frac{1}{2}} \end{array} \xrightarrow{\text{Bob: } |+\rangle} \frac{1}{2} |1\rangle \otimes \underbrace{(|0\rangle - |1\rangle)}_{\text{Bob: } |+\rangle}$$

- In other words, Bob's state collapses to $|+\rangle$, if Alice sees $|0\rangle$ from her H gate and measurement.

Bob's state is immediately altered due to Alice's action?
Is this a faster-than-light communication?

In fact, this time we can prove Einstein wrong...
The reasoning behind this is that:

In any case (regardless of Alice's action), if Bob measures his qubit, he will always observe $|0\rangle$ or $|1\rangle$ with $\frac{1}{2}$ probability. This "paradox" is actually due to the clunkiness of our description of (pure) quantum state. We need a different description of quantum state of Bob's qubit that is not affected by Alice's action. That is called mixed state representation or density operator rep.
More to come in later lectures!