

Lecture 6 CPSL 447/547 - Intro to QC

Measurements

Outline

- Born's Rule
- Partial measurement
- Quantum circuit.

From earlier lecture,

Measuring quantum state $|ψ\rangle = \alpha|0\rangle + \beta|1\rangle$ means:

Collapsing $|ψ\rangle$ to $\begin{cases} |0\rangle \text{ with probability } |\alpha|^2, \\ |1\rangle \text{ with probability } |\beta|^2. \end{cases}$

"Probabilistic" + "Irreversible".

Measurement basis

Measurement is an observation/readout experiment
on qubits along a basis.

E.g. z-basis (default): $\{|0\rangle, |1\rangle\}$

Projectors: $\Pi_0 = |0\rangle\langle 0|$, $\Pi_1 = |1\rangle\langle 1|$

Dimension to describe measurement:

projections -> measure

$$\begin{aligned}\pi_0|\psi\rangle &= |0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha|0\rangle\langle 0| + \beta|0\rangle\langle 1| \\ &= \alpha|0\rangle. \quad \text{"projected/collapsed state"} \\ \pi_1|\psi\rangle &= \beta|1\rangle. \quad (\text{up to normalization})\end{aligned}$$

$$\begin{aligned}\langle\psi|\pi_0|\psi\rangle &= (\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha|0\rangle) = \alpha^*\alpha = |\alpha|^2 \\ \langle\psi|\pi_1|\psi\rangle &= |\beta|^2 \quad \text{"probability"}$$

Measurement:

$$|\psi\rangle \xrightarrow{\text{projects}} \frac{\pi_0|\psi\rangle}{\sqrt{\langle\psi|\pi_0|\psi\rangle}} \quad \text{with prob. } \langle\psi|\pi_0|\psi\rangle$$
$$\xrightarrow{\quad} \frac{\pi_1|\psi\rangle}{\sqrt{\langle\psi|\pi_1|\psi\rangle}} \quad \text{with prob. } \langle\psi|\pi_1|\psi\rangle.$$

Observable

$$\sigma_z = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1| \quad (\text{spectral theorem})$$

$\uparrow \quad \uparrow$
eigenvalues of σ_z .

Random variable

$$z = \begin{cases} +1 & \text{if measured } |0\rangle \text{ : with prob. } \langle\psi|\pi_0|\psi\rangle \\ -1 & \text{if measured } |1\rangle \text{ : with prob. } \langle\psi|\pi_1|\psi\rangle. \end{cases}$$

What is the expectation value of z ?

$$\langle z \rangle = (+1) \cdot \langle\psi|\pi_0|\psi\rangle + (-1) \langle\psi|\pi_1|\psi\rangle$$

$$\begin{aligned}
 &= \langle \psi | ((+1) \pi_0 + (-1) \pi_1) |\psi \rangle \\
 &= \langle \psi | \sigma_z |\psi \rangle.
 \end{aligned}$$

Measurement along a different basis.

Let's try the basis $\{|+\rangle, |-\rangle\}$.

Given $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$

Collapse $|\psi\rangle$ to $\begin{cases} |+\rangle \text{ with prob. } |\alpha|^2 \\ |-\rangle \text{ with prob. } |\beta|^2 \end{cases}$

E.g. $|\psi\rangle = |0\rangle$, what's the measurement outcome?

$$\begin{aligned}
 |\psi\rangle = |0\rangle &= ?|+\rangle + ?|-\rangle \\
 &= \underbrace{\frac{1}{\sqrt{2}}|+\rangle}_{\downarrow} + \underbrace{\frac{1}{\sqrt{2}}|-\rangle}_{\downarrow} \rightarrow \begin{cases} |+\rangle \text{ with prob. } \frac{1}{2} \\ |-\rangle \text{ with prob. } \frac{1}{2} \end{cases} \\
 &\quad |+\rangle + |-\rangle
 \end{aligned}$$

$$|\psi\rangle = \pi_+ |\psi\rangle + \pi_- |\psi\rangle$$

Random variable $x = \begin{cases} +1 & \text{if measured } |+\rangle \\ -1 & \text{if measured } |-\rangle \end{cases}$

$$\langle x \rangle = \langle \psi | \sigma_x | \psi \rangle = \langle \psi | ((+1) \pi_+ + (-1) \pi_-) |\psi \rangle$$

$$\begin{aligned}
 &= (+1) \langle \psi | \pi_+ |\psi \rangle + (-1) \langle \psi | \pi_- |\psi \rangle \\
 &= (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0
 \end{aligned}$$

Born's Rule

Given an observable $O = \sum a_n E_n$ $\xrightarrow{\text{real eigenvalues}}$ $\xrightarrow{\text{orthogonal projectors}}$

where $a_n \in \mathbb{R}$, $E_n E_m = \delta_{nm} E_m$, $E_n = E_n^+$,

We define a measurement (by observable \mathcal{O}):

- Quantum state $|\psi\rangle$ collapses to:

$$|\psi\rangle \longrightarrow \frac{E_n |\psi\rangle}{\sqrt{\langle \psi | E_n | \psi \rangle}}$$

with probability:

$$\langle \psi | E_n | \psi \rangle.$$

- We say the measurement outcome is a random variable $A = a_n$ with prob $\langle \psi | E_n | \psi \rangle$.
- The expectation value of this measurement:

$$\begin{aligned}\langle \mathcal{O} \rangle &= \sum_n a_n \text{Prob}[A=a_n] = \sum_n a_n \langle \psi | E_n | \psi \rangle \\ &= \langle \psi | \left(\sum_n a_n E_n \right) | \psi \rangle \\ &= \langle \psi | \mathcal{O} | \psi \rangle.\end{aligned}$$

Partial Measurement

How to measure entangled qubits?

E.g. Alice (A) Bob (B)



Entangled joint state: $|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.
 cannot be written as $|\Psi_A\rangle \otimes |\Psi_B\rangle$

① What if Bob measures only his qubit (in z basis)?

• Observable? $I \otimes \sigma_z = (+) I \otimes |0\rangle\langle 0| + (-) I \otimes |1\rangle\langle 1|$.
 for A ↑ for B Π_0 Π_1

• $\begin{cases} \text{Prob}(|0\rangle_B) = \langle \Psi_{AB} | \Pi_0 | \Psi_{AB} \rangle = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \\ \text{Prob}(|1\rangle_B) = \langle \Psi_{AB} | \Pi_1 | \Psi_{AB} \rangle = \frac{1}{2} \end{cases}$

② State of A after Bob's measurement?

$\begin{cases} \text{If } |0\rangle_B : \frac{\Pi_0 |\Psi_{AB}\rangle}{\sqrt{1/2}} = |00\rangle = |0\rangle_A \otimes |0\rangle_B \Rightarrow \text{Alice: } |0\rangle_A \\ \text{If } |1\rangle_B : \frac{\Pi_1 |\Psi_{AB}\rangle}{\sqrt{1/2}} = |11\rangle = |1\rangle_A \otimes |1\rangle_B \Rightarrow \text{Alice: } |1\rangle_A. \end{cases}$

③ Alice measures her qubit immediately after:

$\begin{cases} \text{If } |0\rangle_B : \Pr(|0\rangle_A) = 1 & \text{"Correlated outcome!"} \\ \text{If } |1\rangle_B : \Pr(|1\rangle_A) = 1 \end{cases}$

④ $|\Psi_{AB}\rangle$ symmetric, had Alice measured first,

she would get $|0\rangle_A$ or $|1\rangle_A$ with prob. $\frac{1}{2}$ and $\frac{1}{2}$, too.

⑤ What if Bob measures in x-basis, instead of z?

Observables τ_x, τ_y, τ_z

$$U_{\text{entangle}} : \mathbb{I} \otimes U_X$$

We can derive again or observe :

$$\begin{aligned} |\Psi_{AB}\rangle &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}} (|++\rangle + |--\rangle) . \quad \text{Everything follows.} \end{aligned}$$

Alice and Bob would *always* get correlated outcome if they measure in the same basis.

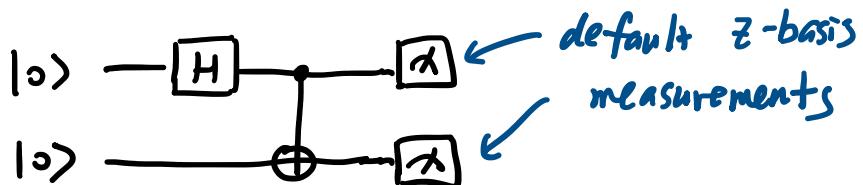
Entangled: Correlated measurement outcome in any "agreed" basis.

Note: Go back to earlier discussion on "EPR paradox".

Quantum Circuits

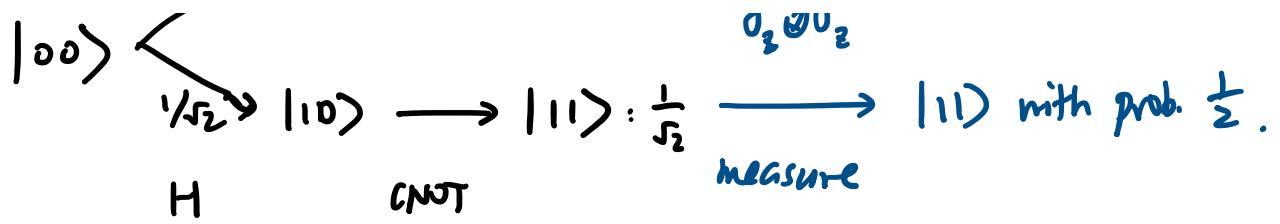
We now have all the elements of quantum circuits:
quantum states, gates, measurements.

Example "Making a Bell pair".



Computational path of the circuit:

$$\frac{1}{\sqrt{2}} \rightarrow |00\rangle \longrightarrow |00\rangle : \frac{1}{\sqrt{2}} \xrightarrow{-\text{---}} |00\rangle \text{ with prob } \frac{1}{2} .$$



Evolving quantum state vector:

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{H \otimes I} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{CNOT}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

More examples in next lecture!