

Lecture 13 CPSC 447/547 - Intro to QC

Quantum Fourier Transform

Outline

- Classical Discrete Fourier Transform (DFT)
- QFT over \mathbb{Z}_2 , \mathbb{Z}_2^n , and \mathbb{Z}_n

So far, in lectures, we have been using $H^{\otimes n}$ for transforming to/from the "Fourier basis"

Can we do more than $H^{\otimes n}$?

In this lecture, we'll see a general framework called, **Quantum Fourier Transform (QFT)**.

(Classical) Discrete Fourier Transform (DFT)

A sequence of complex numbers : $A = \alpha_0, \alpha_1, \dots, \alpha_{N-1}$

\downarrow DFT

Another sequence of complex numbers : $B = \beta_0, \beta_1, \dots, \beta_{N-1}$.

Such transformation (DFT) is **linear**, and **invertible**.
(Also I like it that way)

$$A = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \xrightarrow{\mathcal{F}} B = F_N \cdot A = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{N-1} \end{bmatrix}$$

(vice versa ... for quantum transformations)

F_N : Fourier Matrix.

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{N-1} \\ 1 & w^2 & w^4 & & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & & w^{(N-1)^2} \end{bmatrix}$$

- Column j is powers of w^j
- Entry $F_N[k, j] = w^{jk} / \sqrt{N}$

where $w = e^{i 2\pi / N}$ "N-th root of unity".

Since $B = F_N \cdot A$, by definition,

$$\beta_k = \frac{1}{\sqrt{N}} \sum_j \alpha_j w^{jk}$$

$$\alpha_j = \frac{1}{\sqrt{N}} \sum_k \beta_k w^{jk}.$$

Quantum Fourier Transform :

$$|\psi_A\rangle = A = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = \sum_j \alpha_j |j\rangle, \quad |\psi_B\rangle = B = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1} \end{bmatrix} = \sum_k \beta_k |k\rangle$$

$$[\alpha_{mj}]$$

$$[\beta_{mk}]$$

Fourier Matrix F_N :

Entry at column j , row k : $F_N[j, k] = \frac{1}{\sqrt{N}} W^{jk}$

In other words,

$$F_N = \sum_j \sum_k \frac{1}{\sqrt{N}} W^{jk} |k\rangle X_j |1\rangle.$$

Example 1-qubit . $N=2$.

QFT (over \mathbb{Z}_2)

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \xrightarrow{\mathcal{F}_{\mathbb{Z}_2}} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} . \quad N=2 \quad W = e^{\frac{i2\pi}{2}} = -1$$

$$\Rightarrow F_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{"Hadamard transformation!"}$$

Check:

$$\beta_k = \frac{1}{\sqrt{2}} \sum_j \alpha_j (-1)^{jk} \quad \left\{ \begin{array}{l} \beta_0 = \frac{\alpha_0 + \alpha_1}{\sqrt{2}} \\ \beta_1 = \frac{\alpha_0 - \alpha_1}{\sqrt{2}} \end{array} \right. . \quad \text{☺}$$

Example . n -qubit .

What is $H^{\otimes n}$ in QFT framework?

QFT (over \mathbb{Z}_2^n)

$$\text{Set } \mathcal{F} = F_2^{\otimes n} = H^{\otimes n} , \text{ so } W = e^{\frac{i2\pi}{2}} = -1$$

$$F_2^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{j,k} (-1)^{jk} |k\rangle \langle j|$$

Still, $N=2^n$.

Example, What if we set $N=2^n$ for n qubits.

QFT (over \mathbb{Z}_{2^n})

$$\text{So } N=2^n, w = e^{i\frac{2\pi}{2^n}}$$

$$F_{2^n} = \frac{1}{\sqrt{2^n}} \sum_{j,k} w^{jk} |k\rangle \langle j|.$$

Take $n=4$ qubits as an example.

$$N=2^4=16, w = e^{i\frac{2\pi}{16}} = e^{i\pi/8}.$$

$$F_{16} = \frac{1}{\sqrt{16}} \sum_{j,k} w^{jk} |k\rangle \langle j|$$

Take a bit string $|j\rangle$, What is $F_{16}|j\rangle$?

$$|j\rangle \rightarrow \frac{1}{4} \left(|0\rangle + w^{j \cdot 1} |1\rangle + w^{j \cdot 2} |2\rangle + w^{j \cdot 3} |3\rangle + \dots + w^{j \cdot 15} |15\rangle \right)$$

Question: F_{16} is unitary, can we implement it with a quantum circuit (of 4 qubits)?

Circuit Implementation of QFT

Let's try writing $|k\rangle$ into binary,

$$|j\rangle \xrightarrow{F_{16}} \frac{1}{4} \left(|0000\rangle + w^j |0001\rangle + w^{2j} |0010\rangle + w^{3j} |0011\rangle + \dots + w^{15j} |1111\rangle \right)$$

key observation:

The output state is **unentangled**!

$$\begin{aligned} |j\rangle &\xrightarrow{F_{16}} \frac{1}{\sqrt{2}} (|0\rangle + w^{8j} |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle + w^{4j} |1\rangle) \\ &\otimes \frac{1}{\sqrt{2}} (|0\rangle + w^{2j} |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle + w^j |1\rangle). \end{aligned}$$

So we can build circuit qubit by qubit.

Let $|j\rangle = |j_3 j_2 j_1 j_0\rangle$ as a bit string

$$\Rightarrow j = 8 \cdot j_3 + 4 \cdot j_2 + 2 \cdot j_1 + 1 \cdot j_0 \quad (j = \sum_{s=0}^{n-1} 2^s j_s)$$

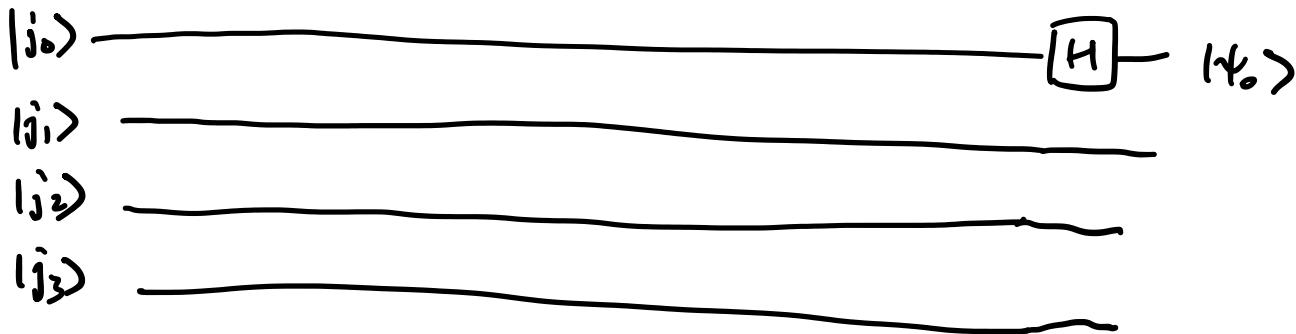
Notice that $w = e^{i\frac{2\pi}{16}}$, so $w^{16} = e^{i2\pi} = 1 = e^{i4\pi}$

$$w^{8j} = e^{i\pi j} = e^{i\pi(\cancel{8j_3} + \cancel{4j_2} + \cancel{2j_1} + 1 \cdot j_0)} = e^{i8\pi} = \dots$$

$$= e^{i\pi j_0} = (-1)^{j_0}$$

$$\Rightarrow \frac{1}{4}(|0\rangle + w^{8j} |1\rangle) = \frac{1}{4} (|0\rangle + e^{i\pi j_0} |1\rangle) = \dots$$

$$|j_0\rangle, |j_1\rangle, |j_2\rangle, |j_3\rangle \rightarrow |\psi_0\rangle$$



$$\text{Next qubit : } |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{\pi}{2}j_1}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{\pi}{2}j_1}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi(\cancel{\frac{4j_3}{2}} + \cancel{\frac{j_2}{2}} + j_1 + \frac{1}{2}j_0)}|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi j_1 + i\frac{\pi}{2}j_0}|1\rangle)$$

First

$$|j_1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi j_1}|1\rangle)$$

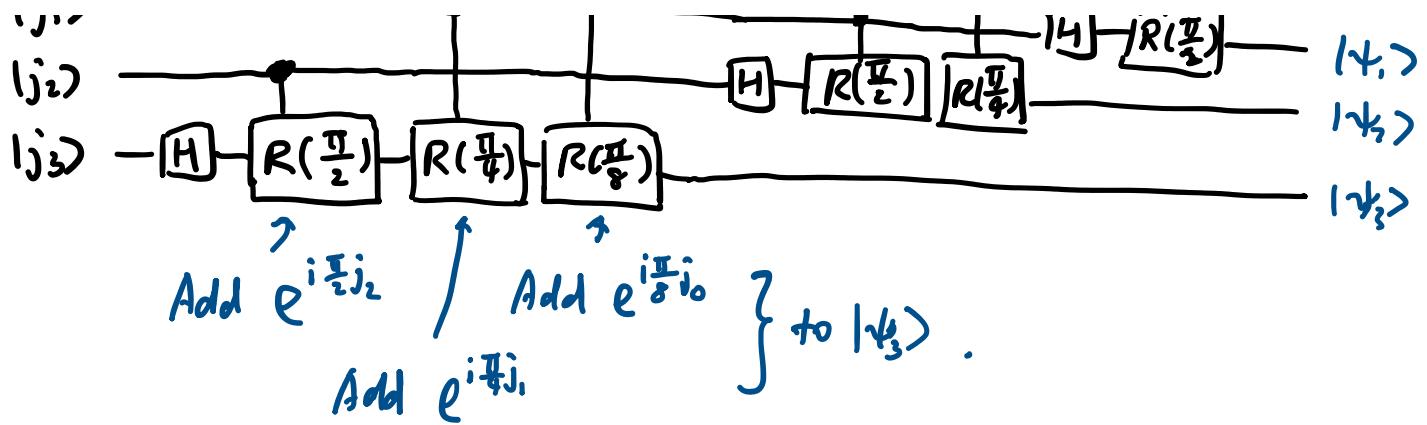
Then.

A quantum circuit diagram for the first qubit $|j_1\rangle$. It starts with a Hadamard gate H , followed by a rectangular box labeled $R(\theta)$. A wavy line connects the output of the H gate to the input of the $R(\theta)$ gate. The output is labeled $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi j_1 + i\theta j_0}|1\rangle)$. A blue label tR_1 is written below the $R(\theta)$ box.

So, set $\theta = \frac{\pi}{2}$, we are done!

Continue for each qubit,





In general, $j = j_{n-1} \dots j_1 j_0 = \sum_{s=0}^{n-1} j_s \cdot 2^s$
 $k = k_{n-1} \dots k_1 k_0 = \sum_{s=0}^{n-1} k_s \cdot 2^s$, $j_s, k_s \in \{0, 1\}$

Transform: $|j\rangle \xrightarrow{F_{2^n}} F_{2^n}|j\rangle = |\psi_0\rangle \otimes |\psi_1\rangle \otimes \dots \otimes |\psi_{n-1}\rangle$

$$F_{2^n}|j\rangle = \frac{1}{\sqrt{2^n}} \sum_k w^{jk} |k\rangle \quad (\text{Here } w = e^{i2\pi/2^n})$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_{n-1} \dots k_1 k_0} w^{j(\sum_s k_s \cdot 2^s)} |k_{n-1} \dots k_1 k_0\rangle$$

$$= \frac{1}{\sqrt{2^n}} \left(\sum_{k_{n-1}} w^{j k_{n-1} \cdot 2^{n-1}} |k_{n-1}\rangle \right) \otimes \dots \otimes \left(\sum_{k_0} w^{j k_0 \cdot 2^0} |k_0\rangle \right)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + w^{j \cdot 2^{n-1}} |1\rangle) \otimes \dots \otimes \frac{1}{\sqrt{2}} (|0\rangle + w^{j \cdot 2^0} |1\rangle)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{ij \frac{2\pi}{2^n}} |1\rangle) \otimes \dots \otimes \frac{1}{\sqrt{2}} (|0\rangle + e^{ij \frac{2\pi}{2^n}} |1\rangle)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i (j_{n-1} \cancel{+ \dots +} j_1 \cancel{+ j_0 \cdot 2^{-1}})} |1\rangle)$$

1'')

$\otimes \dots$

$$\mathfrak{D} \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i (j_{n-1} \cdot 2^{n-1} + \dots + j_1 \cdot 2^1 + j_0 \cdot 2^0)}) |1\rangle)$$

\Rightarrow Construct F_{2^n} qubit by qubit.