

Lecture 16 CPSC 447/547 - Intro to QC

Grover's Search Algorithm II.

Outline

- Review of Grover example.
 - Analysis of amplitude amplification
-

Recall from last lecture,

Unstructured search problem:

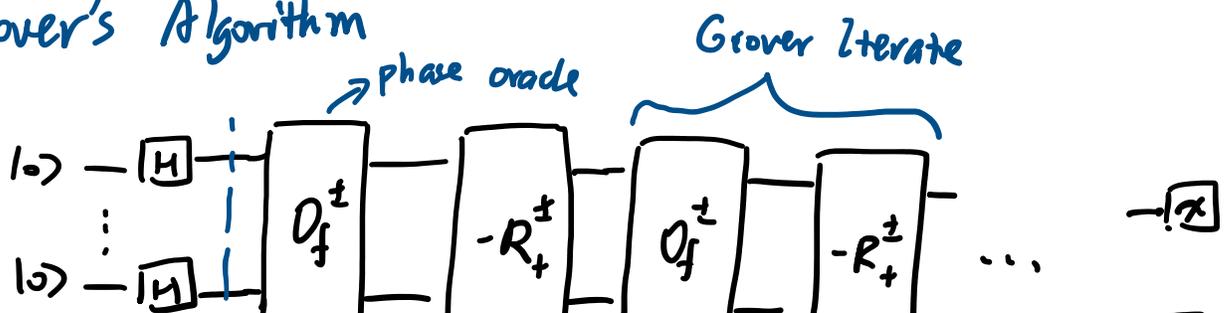
Given oracle access to function $f: \{0,1\}^n \rightarrow \{0,1\}$

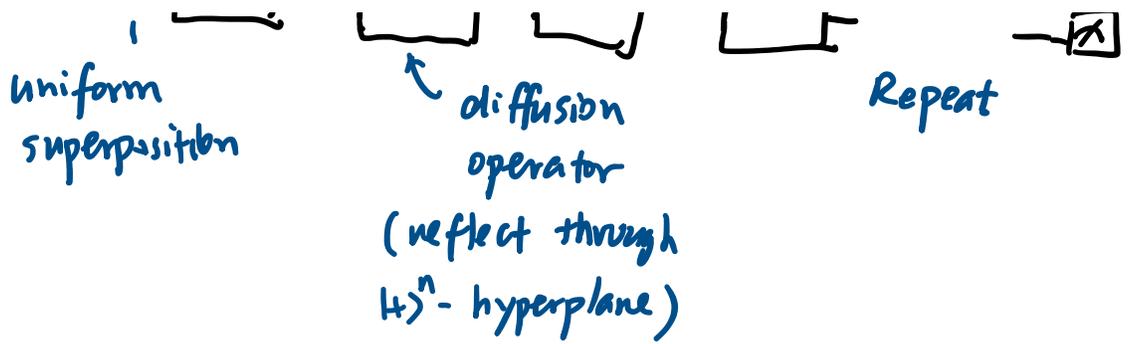
$$f(x) = \begin{cases} 1 & \text{if } x = z \text{ for some } z \in \{0,1\}^n \\ 0 & \text{otherwise} \end{cases}$$

Want to find z , "unique"

later can relax this assumption.

Grover's Algorithm





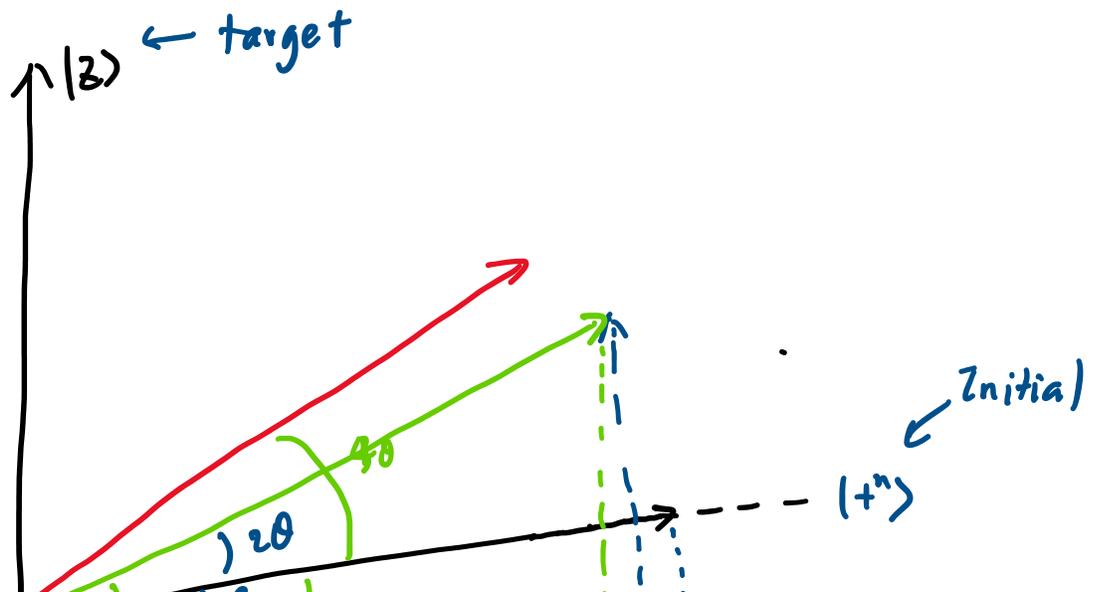
$$O_f^\pm = R_z^\pm = I - 2|z\rangle\langle z|$$

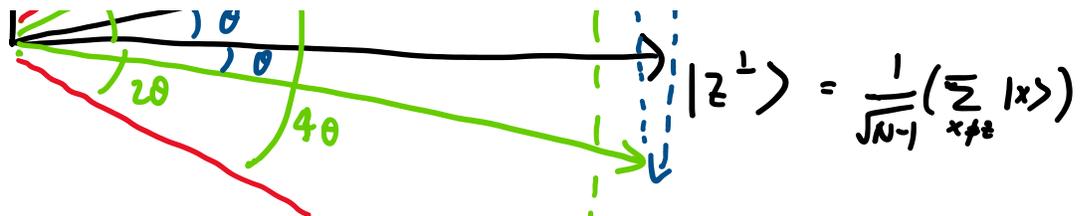
$$-R_+^\pm = 2|t^n\rangle\langle t^n| - I$$

$$\left\{ \begin{array}{l} \sum_x \alpha_x |x\rangle \xrightarrow{O_f^\pm} \sum_{x \neq z} \alpha_x |x\rangle - \alpha_z |z\rangle \quad \text{Reflect through } z \\ \sum_x \alpha_x |x\rangle \xrightarrow{-R_+^\pm} \sum_x (2\mu - \alpha_x) |x\rangle \quad \text{Reflect through } |t^n\rangle \end{array} \right.$$

$\mu = \frac{1}{2^n} \sum_x \alpha_x$: mean of amplitudes

Two Reflections (is a rotation)





$$\langle t^n | z^\perp \rangle = \|t^n\| \cdot \|z^\perp\| \cdot \cos \theta = \cos \theta = \frac{1}{\sqrt{N}} \cdot \frac{1}{\sqrt{N-1}} \cdot (N-1) = \sqrt{\frac{N-1}{N}}$$

Initially,

$$|\psi\rangle = |t^n\rangle = \alpha |z\rangle + \beta |z^\perp\rangle = \sin \theta |z\rangle + \cos \theta |z^\perp\rangle$$

After two reflections

$$|\psi\rangle = \sin(3\theta) |z\rangle + \cos(3\theta) |z^\perp\rangle$$

After two reflections again,

$$|\psi\rangle = \sin(5\theta) |z\rangle + \cos(5\theta) |z^\perp\rangle$$

How many iterations until we reach $|z\rangle$?

$$|\psi\rangle = \sin[(2t+1)\theta] |z\rangle + \cos[(2t+1)\theta] |z^\perp\rangle$$

$$(2t+1)\theta = \frac{\pi}{2} \Rightarrow t = \frac{1}{2} \left(\frac{\pi/2}{\theta} - 1 \right), \quad \theta \approx \frac{1}{\sqrt{N}} \text{ for large } N.$$

$$\approx \frac{\pi}{4} \sqrt{N} = O(\sqrt{N})$$

What if:

- Iterate too many times?

$\sin[(2t+1)\theta]$ starts to go down!

- There are **multiple** x such that $f(x) = 1$?

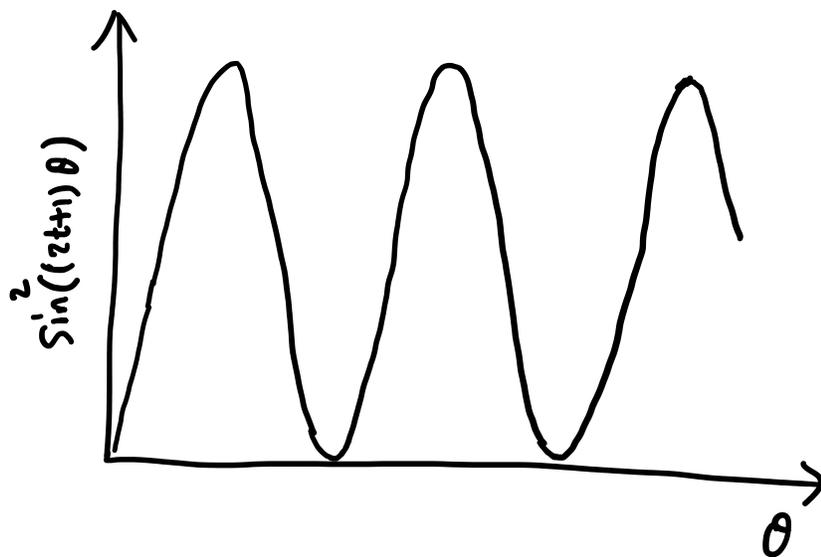
Suppose k items: z_1, z_2, \dots, z_k , $|z\rangle = \frac{1}{\sqrt{k}} \sum_j |z_j\rangle$

$$\langle t^n | z^\perp \rangle = \sqrt{\frac{N-k}{N}} \quad \langle t^n | z \rangle = \sqrt{\frac{k}{N}} = \sin \theta \quad \leftarrow \text{overlap of initial \& target}$$

For $k \ll N$, $\theta \approx \sqrt{\frac{k}{N}}$, $t \approx \frac{\pi}{4} \sqrt{\frac{N}{k}} = O(\sqrt{N/k})$.

- We **do not know how many items** w. $f(x) = 1$?

unknown k , unknown $\sqrt{\frac{N}{k}}$ iterations.



(setting $t = \sqrt{N}$)

Conventional Grover:

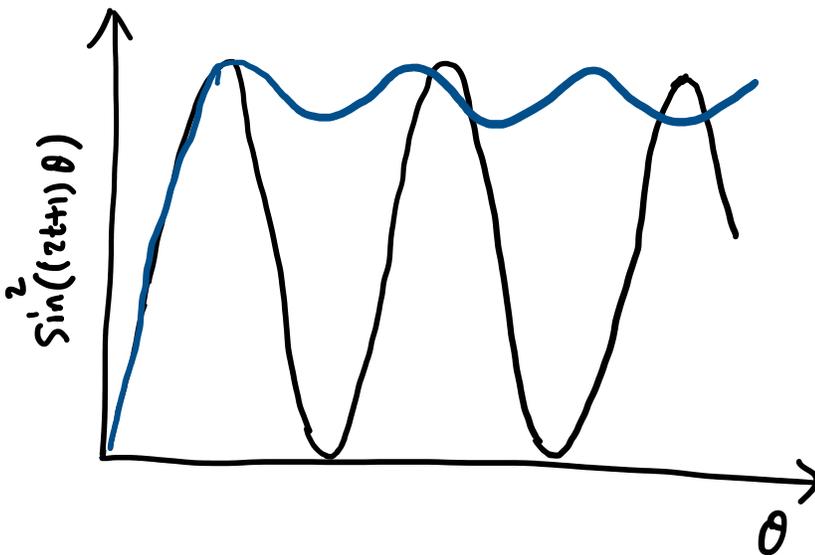
$$\text{Iterate: } -R_t^\pm \cdot O_f^\pm = -(I - 2|t\rangle\langle t|) \cdot (I - 2|z\rangle\langle z|)$$

Fixed-Point Amplitude Amplification:

$$\text{Iterate: } -S_+(\alpha) \cdot S_2(\beta) = -(I - (1 - e^{-i\alpha})(T^+ X T^+)) \cdot (I - (1 - e^{-i\beta})(Z X Z)).$$

S : **Generalized Reflection** Add arbitrary phase to reflection.

$$R_+^{\pm} = S_+(\pm\pi), \quad O_f^{\pm} = S_2(\pm\pi).$$



- Fixed-Point AA
- Conventional AA.

[Yoder, Low, Chuang '14]