

# Lecture 17 CPSC 447/547 - Intro to QC

## Noisy Quantum States & Density Operators.

### Outline

- Quantum states as density operators
- QC Principles, revisited.
- Properties of density matrices.

Today, we are going to study a more general form of quantum states, called **Mixed States**.

This is the general form we use to study :

**noisy quantum states**, **open quantum systems**,  
**or probability distribution of quantum states**, etc.

### Scenario 1 :

We built a prototype QC. When we initialize a qubit to the  $|1\rangle$  state, we end up in

$$|\psi\rangle = \begin{cases} |0\rangle \text{ with prob. } \varepsilon \\ |1\rangle \text{ with prob. } 1-\varepsilon \end{cases}$$

due to **imprecision in the controls**.

## Scenario 2:

We want to initialize a qubit to the  $|1\rangle$  state, but end up entangling with environment (another qubit)

$$|\psi_{\text{sys-env}}\rangle = \sqrt{\varepsilon} |00\rangle + \sqrt{1-\varepsilon} |11\rangle$$

due to interaction with environment.

Note: we are not sure whether environment qubit has decohered (measured). If measured,

$$|\psi_{\text{sys}}\rangle = \begin{cases} |0\rangle \text{ with prob. } \varepsilon \\ |1\rangle \text{ with prob. } 1-\varepsilon. \end{cases}$$

## Scenario 3:

You went to a qubit store, two qubits are on sale:

$$|\psi_1\rangle = \begin{cases} |0\rangle \text{ with prob. } \frac{1}{2} \\ |1\rangle \text{ with prob. } \frac{1}{2} \end{cases} \quad |\psi_2\rangle = \begin{cases} |+\rangle \text{ with prob. } \frac{1}{2} \\ |- \rangle \text{ with prob. } \frac{1}{2} \end{cases}$$

You want to know which has more "value".

Note: we will soon see that  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are identical.

## Ensemble of Quantum States ("Mixed State")

Suppose  $|\psi\rangle$  is  $|\phi_j\rangle$  with prob.  $p_j$  for  $j=0,1,2,\dots$ .

What is the measurement outcome distribution?

Observable :  $O = (+) \underbrace{\Pi_0}_{1 \times 1} + (-) \underbrace{\Pi_1}_{1 \times 1}$  projectors

Prob. of "10"

$$\begin{aligned} & \sum_j p_j \cdot \langle \phi_j | \Pi_0 | \phi_j \rangle \\ &= \sum_j p_j \langle \phi_j | \underbrace{0 \times 1}_{\text{X}} | \phi_j \rangle \\ &= \sum_j p_j \langle \phi_j | \underbrace{0 \times \phi_j}_0 | \phi_j \rangle \\ &= \langle 0 | \left( \sum_j p_j | \phi_j \times \phi_j | \right) | 0 \rangle \\ &= \langle 0 | \rho | 0 \rangle \end{aligned}$$

Let  $\rho = \sum_j p_j | \phi_j \times \phi_j |$ ,  
then only  $\rho$  depends  
on the state.  
 $\rho$  is a matrix for all info about the state.

Expectation

$$\begin{aligned} & \sum_j p_j \langle \phi_j | O | \phi_j \rangle \\ &= \sum_j p_j \text{tr}(O | \phi_j \times \phi_j |) \\ &= \text{tr}(O \cdot \sum_j p_j | \phi_j \times \phi_j |) \\ &= \text{tr}(O \cdot \rho) \end{aligned}$$

$$\langle \psi | \phi \rangle = \text{tr}(|\psi \times \phi|)$$

Quantum Gates  $| \phi_j \rangle \rightarrow U | \phi_j \rangle$ .

$$\begin{aligned} \rho &= \sum_j p_j | \phi_j \times \phi_j | \longrightarrow \sum_j p_j U | \phi_j \times \phi_j | U^+ \\ &= U \left( \sum_j p_j | \phi_j \times \phi_j | \right) U^+ = U \rho U^+ \end{aligned}$$

Quantum State as Density Operator.

$$P = \sum_j p_j |\psi_j \times \phi_j| \leftarrow \text{convenient representation for ensemble of } g \text{ states.}$$

Also called, density matrix.

with prob. density  $p_j \geq 0$ ,  $\sum_j p_j = 1$ .

Example .

$$|\Psi_1\rangle = \begin{cases} |0\rangle \text{ w.p. } 1/2 \\ |1\rangle \text{ w.p. } 1/2 \end{cases}, \quad |\Psi_2\rangle = \begin{cases} |+\rangle \text{ w.p. } 1/2 \\ |- \rangle \text{ w.p. } 1/2 \end{cases}$$

$$P_1 = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| \quad P_2 = \frac{1}{2} |+\rangle\langle +| + \frac{1}{2} |- \rangle\langle -|$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$P_1 = P_2 : \text{Same!}$$

**Same state**, multiple ways of writing as an ensemble.

In fact, infinite ways:  $P_3 = \frac{1}{2} |+i\rangle\langle +i| + \frac{1}{2} |-i\rangle\langle -i|$ , etc.

To see this, we turn to Bloch Sphere again!

### Density Operator in Bloch Sphere

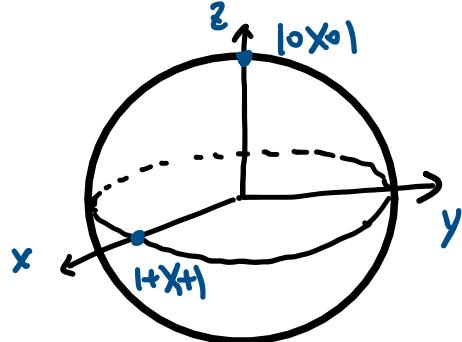
$|\psi\rangle = \cos\frac{\theta}{2} |0\rangle + e^{i\phi} \sin\frac{\theta}{2} |1\rangle$ , Generalize to density operator:

$$P = |\psi \times \psi| = \frac{1}{2} (I + r_x \cdot \vec{\sigma}_x + r_y \cdot \vec{\sigma}_y + r_z \cdot \vec{\sigma}_z) = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma})$$

$\downarrow \quad \uparrow \quad \uparrow$   
Pauli matrices  $r_x^2 + r_y^2 + r_z^2 = 1$ .

$(r_x, r_y, r_z)$  is the Cartesian coordinate of P.

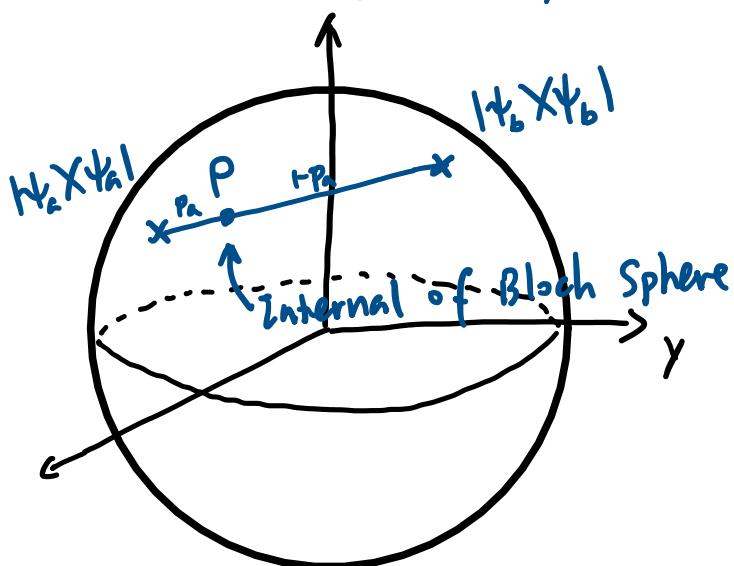
Example:  $|0\rangle\langle 0| = \frac{1}{2}(I + \sigma_z) : (0, 0, 1)$ .



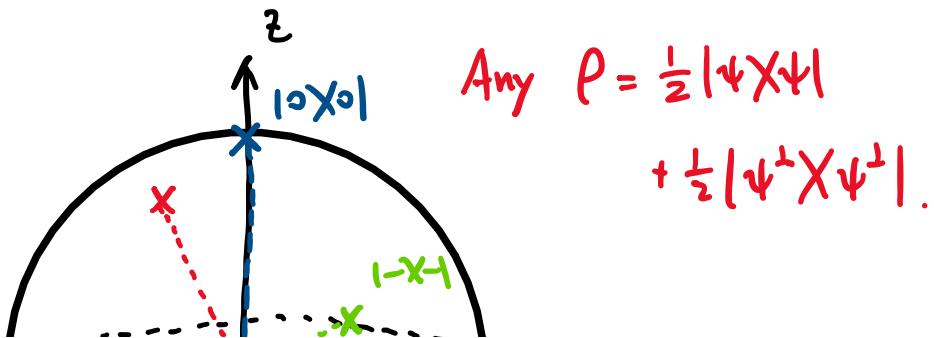
Pure states,  $|\vec{r}|^2 = 1$   
(on surface of sphere).

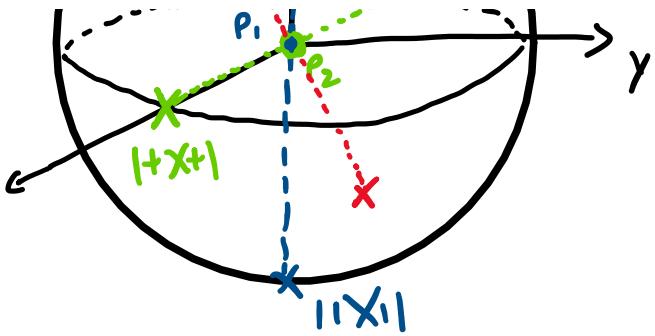
Bloch Sphere rep for mixed states. ( $r_x^2 + r_y^2 + r_z^2 \leq 1$ )

$\rho = p_a |\psi_a \rangle \langle \psi_a| + (1-p_a) |\psi_b \rangle \langle \psi_b|$  (convex combination of pure states:  $|\psi_a \rangle \langle \psi_a|, |\psi_b \rangle \langle \psi_b|$ .)



$$\rho_1 = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| . \quad \rho_2 = \frac{1}{2} |+\rangle\langle +| + \frac{1}{2} |- \rangle\langle -| .$$





Given  $P$  inside of the Bloch sphere, we can find infinite ways to decompose as ensemble.

But, if  $P$  is not at origin, then there's a

unique ensemble, s.t.  $P = p|\psi\rangle\langle\psi| + (1-p)|\psi^\perp\rangle\langle\psi^\perp|$

where  $\langle\psi|\psi^\perp\rangle = 0$ . "orthogonal".

"Unique basis": line goes through origin and  $P$ .

## Properties of Density Operators

$$\rho = \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

① Because  $(|\phi_j\rangle\langle\phi_j|)^+ = |\phi_j\rangle\langle\phi_j|$

$$(\sum_j p_j |\phi_j\rangle\langle\phi_j|)^+ = \sum_j p_j |\phi_j\rangle\langle\phi_j| \quad \text{"Symmetric"}$$

$$\rho^+ = \rho \quad \text{Hermitian.}$$

② Because measurement probability must be  $\geq 0$ .

"Positive"

$\langle u | \rho | u \rangle \geq 0$ , for any basis  $|u\rangle$ .

Positive Semi-definite, i.e.  $\rho \succcurlyeq 0$ .

③ Because  $\sum_j p_j = 1$ , and  $\langle \phi_j | \phi_j \rangle = 1$ .

$$\begin{aligned} \text{tr}(\rho) &= \text{tr}\left(\sum_j p_j |\phi_j\rangle\langle\phi_j|\right) && \text{"Unity"} \\ &= \sum_j p_j \text{tr}(|\phi_j\rangle\langle\phi_j|) \\ &= \sum_j p_j \langle\phi_j|\phi_j\rangle = 1 \end{aligned}$$

Unit trace, i.e.  $\text{tr}(\rho) = 1$ .

So, a density matrix for  $n$  qubits is a

Hermitian matrix  $\rho \in \mathbb{C}^{2^n \times 2^n}$ , where  $\rho \succcurlyeq 0$ ,  $\text{tr}(\rho) = 1$ .

(Compare with probability density:

A probability density on  $k$  outcomes is a vector  $p \in \mathbb{R}^k$  with  $p \geq 0$  and  $\sum_i p_i = 1$ . )  
each element

Next time, we will use density matrices to describe noisy q. states and noisy processes.