Lecture 18 CPSC 447/547 - Intro to QC.

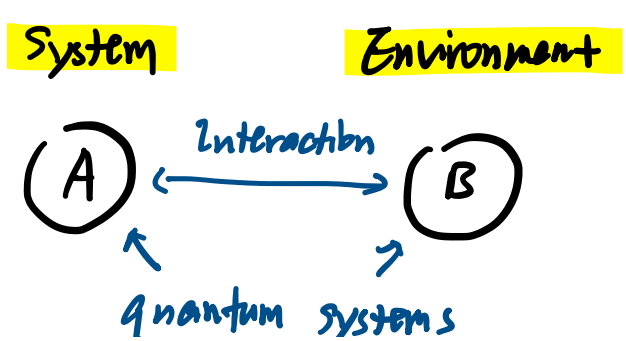
Error Channels

- Reduced Density Operator
- Superoperators
- Operator Sum Representation
- Noise channels.

In last lecture, we introduced Density Operator ρ to represent the state of a quantum system. This allows us to understand the behavior of open quantum systems

Open Quantum Systems (System - Environment model)

Simple model:



In effect, we can assume that system A and environment B starts out in a (pure) product state:

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

then interaction happens between A and B.

Assume entangling unitary transformation $|\psi'_{AB}\rangle = U|\psi_{AB}\rangle$

Question for today, what is the effect of this sys-env interaction on A?

Reduced Density Operator

Example Suppose $|\psi'_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

(Both sys and env are single-qubit states)

What if we measure system A? Partial measurement!

We write the observable $O_A \otimes I_B$.

Claim: For any sys-env (pure) state $|\psi'_{AB}\rangle$,

there exists a product (mixed) state $\rho_A \otimes \rho_B$,

such that $O_A \otimes I_B$ yield the same outcomes.

Let's see what is this ρ_A for $|\psi'_{AB}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

For any observable $O_A \otimes I_B$, the expectation value:

$$\langle O_A \otimes I_B \rangle = \langle \psi_{AB} | O_A \otimes I_B | \psi_{AB} \rangle$$

$$= \frac{1}{2} \langle 0 | O_A | 0 \rangle \langle 0 | I_B | 0 \rangle$$

$$+ \frac{1}{2} \langle 0 | O_A | 1 \rangle \cancel{\langle 0 | I_B | 1 \rangle}^{\,0}$$

$$+ \frac{1}{2} \langle 1 | O_A | 0 \rangle \cancel{\langle 1 | I_B | 0 \rangle}^{\,0}$$

$$+ \frac{1}{2} \langle 1 | O_A | 1 \rangle \langle 1 | I_B | 1 \rangle$$

$$O_A = \begin{bmatrix} \langle 0 | O_A | 0 \rangle & \langle 0 | O_A | 1 \rangle \\ \langle 1 | O_A | 0 \rangle & \langle 1 | O_A | 1 \rangle \end{bmatrix}$$

$$= \underbrace{\frac{1}{2} \langle 0 |}\underbrace{O_A | 0 \rangle} + \underbrace{\frac{1}{2} \langle 1 |}\underbrace{O_A | 1 \rangle}.$$

Only diagonal entries of O_A

$$= \mathrm{Tr}\left(O_A \cdot \tfrac{1}{2} |0\rangle\langle 0| \right) + \mathrm{Tr}\left(O_A \cdot \tfrac{1}{2} |1\rangle\langle 1| \right)$$

$$= \mathrm{Tr}\Big(O_A \cdot \underbrace{\left(\tfrac{1}{2} |0\rangle\langle 0| + \tfrac{1}{2} |1\rangle\langle 1| \right)}_{\rho_A} \Big)$$

Let's generalize.

$$|\psi_{AB}\rangle = \sum_{a,b} \alpha_{ab} |a\rangle_A \otimes |b\rangle_B$$

$\{|a\rangle\}$: basis for sys A

$\{|b\rangle\}$: basis for env B.

$$\langle \psi_{AB} | O_A \otimes I_B | \psi_{AB} \rangle = \left(\sum_{c,d} \alpha^*_{cd} \langle c|_A \otimes \langle d|_B \right) (O_A \otimes I_B) \left(\sum_{a,b} \alpha_{ab} |a\rangle_A \otimes |b\rangle_B \right)$$

$\langle d | b \rangle = \delta_{bd}$

$$= \sum_{b,a,c} \alpha^*_{cb} \alpha_{ab} \langle c | O_A | a \rangle$$

$$= \mathrm{Tr}(O_A \rho_A)$$

where $\rho_A = \sum_{b,a,c} \alpha_{cb}^* \alpha_{ab} |a\rangle\langle c| \equiv Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$

Reduced density operator "Partial trace".

$\rho_A = Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$ "Trace out B from joint state"

<u>This can also be viewed as</u>:

Starting at $|\psi_{AB}\rangle\langle\psi_{AB}|$, imagine we measure env B. in basis $\{|b\rangle_B\}$. but ignoring the measurement outcome, so we have to average over all possible post-measurement states:

- O_B as projectors $\{M_b = I \otimes |b\rangle\langle b|\}$, $\sum_b M_b^\dagger M_b = I$ to preserve norm.
- We obtain the post-measurement state for sys A:

$$|\psi_b\rangle = \frac{M_b|\psi_{AB}\rangle}{\|M_b|\psi_{AB}\rangle\|} \quad \text{with prob. } Prob(b) = \|M_b|\psi_{AB}\rangle\|^2$$

$$\Rightarrow \rho_A = \sum_b Prob(b)\, |\psi_b\rangle\langle\psi_b| = \sum_b M_b|\psi_{AB}\rangle\langle\psi_{AB}|M_b^\dagger$$

$$= \sum_b \left(\sum_a \alpha_{ab}|a\rangle\right)\left(\sum_c \alpha_{cb}^* \langle c|\right) = Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$$

<u>Superoperators / Quantum Channels</u>

How do we describe the above evolution in general?

Initial density operator $\xrightarrow[\text{to}]{\text{maps}}$ Final density operator

$\rho = |\psi_{AB}\rangle\langle\psi_{AB}|$ for system AB

$\mathcal{E}(\rho) = \rho_A = \sum_b M_b \rho M_b^\dagger$ for system A.

A **quantum channel** is a linear map that takes density operators to density operators.

So what's a valid quantum channel $\mathcal{E}(\rho)$?

① Linear

$$\mathcal{E}(a\rho_a + b\rho_b) = a\mathcal{E}(\rho_a) + b\mathcal{E}(\rho_b)$$

due to linearity of density operators.

② Preserve Hermitian property of ρ

If $\rho = \rho^\dagger$, then $\mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger$

because density operators are Hermitian operators.

③ Preserve positive property of ρ

If $\rho \geq 0$, then $\mathcal{E}(\rho) \geq 0$.

④ Preserve trace property of ρ.

$$\mathrm{Tr}(\rho) = \mathrm{Tr}(\mathcal{E}(\rho))$$

In summary, we call a quantum channel

a completely* positive trace preserving (CPTP) map.

(*) "Completely": $\mathcal{E} \otimes I_n$ always positive when we extend $\mathcal{E}$ to a map acting on larger system (with identity map on n-dim anilla coupled to sys).

Operator Sum Representation.

Just like the earlier example, we can mathematically express a quantum channel, $\mathcal{E}(\rho)$, as

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger .$$

where the operation elements E_k are called Kraus Operators, satisfying $\sum_k E_k^\dagger E_k = I$.