

# Lecture 22 (PSC 447/547) - Intro to QC

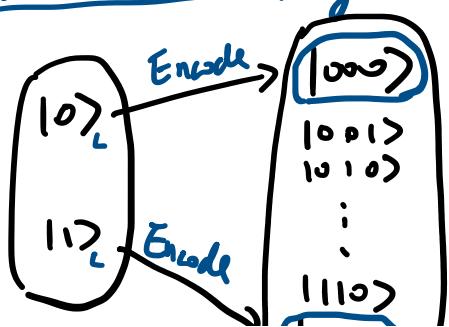
## Stabilizer Formalism

### Outline

- Correcting phase flips
- 9-qubit Shor code
- Stabilizer code
- 7-qubit Steane code.

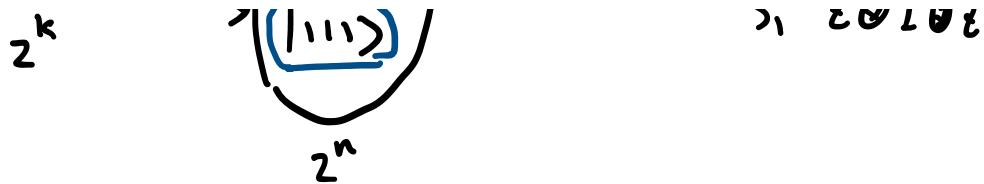
So far in lectures, we have primarily focused on correcting **bit flip errors** in qubits. This lecture introduces how to also correct **phase flips** and generalizes to the **Stabilizer Formalism**.

### Review: Correcting Bit Flips (Use $n=3$ bits to protect $k=1$ bit of info)



Codewords:  $x_0 = |000\rangle, x_1 = |111\rangle$ .

Stabilizer operators:  
 $S_0 = Z \otimes Z \otimes I$   
 $S_1 = Z \otimes I \otimes Z$



Encoded state:  $|\Psi_L\rangle = \alpha|0\rangle_L + \beta|1\rangle_L = \alpha|000\rangle + \beta|111\rangle$

Errors:  $|b\rangle \rightarrow X|b\rangle$  acting on one of physical qubits.  
 $= |b+1\rangle$  (addition modulo 2)

$E|\Psi_L\rangle = \alpha|000+\epsilon\rangle + \beta|111+\epsilon\rangle$ , for some  $\epsilon$  with weight 1.

Diagnose (Stabilizer checks):  $\text{wt}(\epsilon) = \# \text{ of } 1\text{'s in } \epsilon = 1$ .

$|000\rangle, |111\rangle$  are (+1)-eigenstates of  $S_0$  and  $S_1$ .

but  $|000+\epsilon\rangle, |111+\epsilon\rangle$  might be (-1)-eigenstates of  $S_0$  or  $S_1$ .

For example, if  $S_0 = -I$ ,  $S_1 = I$ , then infer  $\epsilon = 0|0\rangle$

We can correct  $\epsilon$  by flipping second qubit back.

## How to Correct Phase Flips

Observation: what if we apply qubit-wise Hadamard on codewords?

$$\begin{aligned} |000\rangle &\xrightarrow{H^{\otimes 3}} |+++ \rangle = |0\rangle_L && \text{new codewords} \\ |111\rangle &\xrightarrow{H^{\otimes 3}} |--- \rangle = |1\rangle_L \end{aligned}$$

Encoded state:  $|\Psi_L\rangle = \alpha|0\rangle_L + \beta|1\rangle_L = \alpha|+++ \rangle + \beta|--- \rangle$ .

Summary:  $|b\rangle \rightarrow Z|b\rangle = (-1)^b |b\rangle$ , for  $b=0,1$ .

Or in phase basis  $\{|0_p\rangle = |+\rangle, |1_p\rangle = |-\rangle\}$

$$|0_p\rangle \rightarrow Z|0_p\rangle = |1_p\rangle$$

$$|1_p\rangle \rightarrow Z|1_p\rangle = |0_p\rangle.$$

$$|\psi_i\rangle = \alpha|000_p\rangle + \beta|111_p\rangle$$

$$E|\psi_i\rangle = \alpha|000_p + e_p\rangle + \beta|111_p + e_p\rangle.$$

Diagnose: New stabilizers:  $S_0' = X \otimes X \otimes I$ ,  $S_1' = X \otimes I \otimes X$ .

Everything follows similarly.

If  $S_0' = -I$ ,  $S_1' = I$ , we can infer  $e_p = \omega|0_p\rangle$   
"phase flip on qubit 2".

How to Correct Both Bit Flips and Phase Flips?

"Concatenation"

9-Qubit Shor Code ( $n=9, k=1$ )

$$\text{Codewords: } |0\rangle_L = \left( \frac{|000\rangle + |111\rangle}{\sqrt{2}} \right)^{\otimes 3}$$

Three clusters of  
three qubits

$$|1\rangle_L = \left( \frac{|000\rangle - |111\rangle}{\sqrt{2}} \right)^{\otimes 3}.$$

Stabilizers:  $Z_1 Z_2 = Z \otimes Z \otimes I \otimes I \otimes I \otimes \cdots \otimes I$

$$Z_1 Z_3$$

$$Z_4 Z_5$$

$$Z_6 Z_7$$

$$Z_8 Z_9$$

$$\begin{array}{c} \dots \\ Z_1, Z_2 \\ X_1, X_2, X_3, X_4, X_5, X_6 \\ X_7, X_8, X_9 \end{array}$$

Diagnose:

- $|0\rangle_L, |1\rangle_L$  are  $(+1)$ -eigenstates of all stabilizers?
- Can we identify a single-qubit error if we report  $\pm 1$  for each syndrome?
  - ① Bit flip ( $X$ )
  - ② Phase flip ( $Z$ )
  - ③ Both ( $Y = iXZ$ )

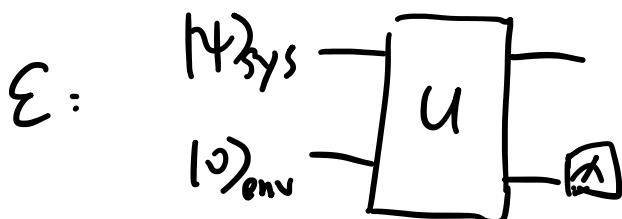
(Note: Shor code is "degenerate":  $Z_1|\Psi\rangle = Z_2|\Psi\rangle$ .)

$Z_1, Z_2$  affects the codewords the same way.)

Pauli Operators are a Basis.

Error channel (in **sys-env model**)

Initial:  $|\Psi_{sys}\rangle = \alpha|0\rangle + \beta|1\rangle$ . env:  $|0\rangle_{env}$



Suppose **unitary  $U$** :

$$U|000\dots000\rangle = \alpha|000\dots000\rangle + \beta|111\dots111\rangle$$

$$U(\alpha|0\rangle + \beta|1\rangle) |0\rangle_{\text{env}} = U(|0\rangle_{\text{env}} \otimes |0\rangle_{\text{env}} + |1\rangle_{\text{env}} \otimes |1\rangle_{\text{env}})$$

$$|\psi\rangle + \beta(|0\rangle |\phi_{10}\rangle_{\text{env}} + |1\rangle |\phi_{11}\rangle_{\text{env}})$$

$$= (\underbrace{\alpha|0\rangle + \beta|1\rangle}_{\text{state } |\psi\rangle} \otimes \underbrace{\frac{1}{2}(|\phi_{00}\rangle + |\phi_{10}\rangle)}_{\text{state } |\Phi_I\rangle})$$

$$X|\psi\rangle \rightarrow +(\alpha|1\rangle + \beta|0\rangle) \otimes \frac{1}{2}(|\phi_{01}\rangle + |\phi_{10}\rangle) \rightarrow |\Phi_X\rangle$$

$$iY|\psi\rangle \rightarrow +(\alpha|1\rangle - \beta|0\rangle) \otimes \frac{1}{2}(|\phi_{01}\rangle - |\phi_{10}\rangle) \rightarrow |\Phi_Y\rangle$$

$$+ (\underbrace{\alpha|0\rangle - \beta|1\rangle}_{\text{state } Z|\psi\rangle} \otimes \frac{1}{2}(|\phi_{00}\rangle - |\phi_{11}\rangle)) \rightarrow |\Phi_Z\rangle.$$

$$= I|\psi\rangle|\Phi_I\rangle + X|\psi\rangle|\Phi_X\rangle + iY|\psi\rangle|\Phi_Y\rangle + Z|\psi\rangle|\Phi_Z\rangle.$$

Next we measure environment (trace out  $|\Phi\rangle$ ) .

So.  $U$  can be viewed as an expansion of four

possible outcomes :  $\begin{cases} Z|\psi\rangle \text{ (no error)} \\ X|\psi\rangle \text{ (Bit flip error)} \\ iY|\psi\rangle \text{ (Both errors : } Y=iXZ\text{)} \\ Z|\psi\rangle \text{ (Phase flip error)} \end{cases}$

Remark :  $|\Phi_I\rangle, |\Phi_X\rangle, |\Phi_Y\rangle, |\Phi_Z\rangle$  need not be orthogonal.

But it's ok to view the error as a probability distribution of the four outcomes.

For n-qubit system. we can expand in terms of

$\{I, X, iY, Z\}^{\otimes n}$  "Pauli strings".

E.g. Error  $\Sigma$  is a subset of all Pauli strings.

$$\Sigma \subseteq \{I, X, iY, Z\}^{\otimes n}.$$

## Stabilizer Code

Define **stabilizer group**:  $P_n = \{\pm I, \pm X, \pm iY, \pm Z\}^{\otimes n}$

Use  $P_n$  to construct a quantum code:

① Pick **stabilizer operators**:  $S \subseteq P_n$ .

② Resulting **codewords**:

$$C_S = \{|\psi\rangle : g|\psi\rangle = |\psi\rangle, \forall g \in S\}$$

"Codewords are **simultaneous** ( $\pm$ ) eigenstates

of all operators in the set  $S$ ".

Remark: ① Require  $-I \notin S$ . Why? ( $|\psi\rangle = 0$ )

②  $\forall g, h \in S, gh = hg$  "Commute". Why?

(note that Pauli's commute or anti-commute  
so  $gh = -hg \Rightarrow |\psi\rangle = 0$ ).

Example (Repetition code)

$$S = \{III, ZIZ, ZIZ, IZZ\} = \langle ZIZ, ZIZ \rangle$$

(independent) generators

$$C_S = \{|000\rangle, |111\rangle\}.$$

What errors can be detected?

$$\mathcal{E} \subseteq P_n.$$

- Single-qubit bit flips :  $\mathcal{E} = \{XII, IXI, IIX\}$

$$|000\rangle \xrightarrow{\mathcal{E}} XIZ|000\rangle = |100\rangle \xrightarrow{S} \langle ZIZ, ZIZ \rangle$$

 Can be diagnosed!

$$(ZIZ \cdot XIZ|000\rangle = -|000\rangle)$$

anti-commute

- Two-qubit bit flips :  $\mathcal{E} = \{XXI, XZX, ZXZ\}$ .

$$\underbrace{ZIZ \cdot XXI}_{\text{commute}}|000\rangle = |000\rangle \quad \text{so} \quad \langle +1, -1 \rangle$$

always

 Cannot be diagnosed.

Cannot distinguish from ZXZ.

7-qubit Steane Code. ( $n=7, k=1$ )

(constructed from classical Hamming code)

Next time!