

Lecture 23 CPSC 447/547 - Intro to QC

Stabilizer codes and Normalizers

Outline

- Stabilizer group
- Steane Code
- Logical Pauli Operations

Review : $P_n = \{\pm I, \pm X, \pm Y, \pm Z\}^{\otimes n}$

Stabilizer group : $S \subseteq P_n$.

Example : (For 3-bit repetition code)

$$S = \{Z\bar{Z}I, ZI\bar{Z}, I\bar{Z}\bar{Z}, III\}$$

= $\langle Z\bar{Z}I, ZI\bar{Z} \rangle$ generators.

Condition : ① $-I \notin S$. ② $\forall g, h \in S, gh = hg$

Codewords:

$$C_S = \{|\psi\rangle : g|\psi\rangle = |\psi\rangle, \forall g \in S\}$$

"Simultaneous (+1)-eigenstates".

Example :

$$\rightarrow C_3 = \{ |000\rangle, |111\rangle \}$$

Error: S can diagnose $\Sigma = \{X11, IXI, IIX\}$

↳ Each syndrome can identify $e \in \Sigma$.

S cannot diagnose $\Sigma' = \Sigma \cup \{XX1, XIX, IXX\}$

↳ because $X11$ and IIX have same syndrome.

Error Correction Condition

A family of errors Σ is **correctable** by code C if and only if,

\forall codewords $|\psi_1\rangle, |\psi_2\rangle \in C$ and $\langle \psi_1 | \psi_2 \rangle = 0$

\forall two errors $E_1, E_2 \in \Sigma$,
 then $\langle \psi_2 | E_2^+ E_1 | \psi_1 \rangle = 0$

"Otherwise can no longer distinguish two codewords".

Example: $|000\rangle \xrightarrow{E_1=X11} |100\rangle \quad |111\rangle \xrightarrow{E_2=IXX} |100\rangle \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{decide would fail.}$

7-Qubit Steane Code ($n=7$, $k=1$)

↳ Can diagnose **any single-qubit errors**

→ construction from classical Hamming Code ($n=7, k=4$)

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Codewords : $x_j = G^T v_j$, $x = \begin{bmatrix} x_0 & x_1 & \dots \end{bmatrix}$, $v = \begin{bmatrix} v_0 & v_1 & \dots \end{bmatrix}$

$$x = G^T v.$$

$$\begin{array}{c} X \\ \downarrow \\ \begin{bmatrix} 1 & | & | & | \\ x_0 & x_1 & \dots & x_{15} \\ 1 & | & | & | \end{bmatrix} \end{array} \quad \begin{array}{c} G^T \\ \downarrow \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c} V \\ \downarrow \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \end{bmatrix}$$

↓
opposite

From Hamming code, we can correct single bit flips.

↓
Quantum code (for correcting bit flips)

$$H \rightarrow S = \begin{bmatrix} Z & Z & Z & Z & 1 & 1 & 1 \\ Z & Z & 1 & 1 & Z & Z & 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow S_0 \\ \leftarrow S_1 \end{array} \quad \text{rows of } S \text{ are operators}$$

$\{z|z\rangle, \bar{z}|\bar{z}\rangle\}$... -> up to ... operations
 S_2 for projective measurements.

$|x_0\rangle, |x_1\rangle, \dots, |x_5\rangle$ are **(+1)-eigenstates** of S_0, S_1, S_2 .

S_0 is any superposition: $|\psi\rangle = \sum_j \alpha_j |x_j\rangle$

$$\Rightarrow S_0 |\psi\rangle = S_1 |\psi\rangle = S_2 |\psi\rangle = |\psi\rangle.$$

Can we use **X-type stabilizers** to check for phase flips?

Let's use the same H :

$$H \longrightarrow S' = \begin{bmatrix} XXXX111 \\ XX11XX1 \\ X1X1X1X \end{bmatrix} \quad \begin{array}{l} S'_0 \\ S'_1 \\ S'_2 \end{array} \quad S_i, S'_j \text{ commute. } \checkmark$$

Are there codewords that are simultaneous **(+1)-eigenstates** for $S_0, S_1, S_2, S'_0, S'_1, S'_2$?

To answer these, we look at two things:

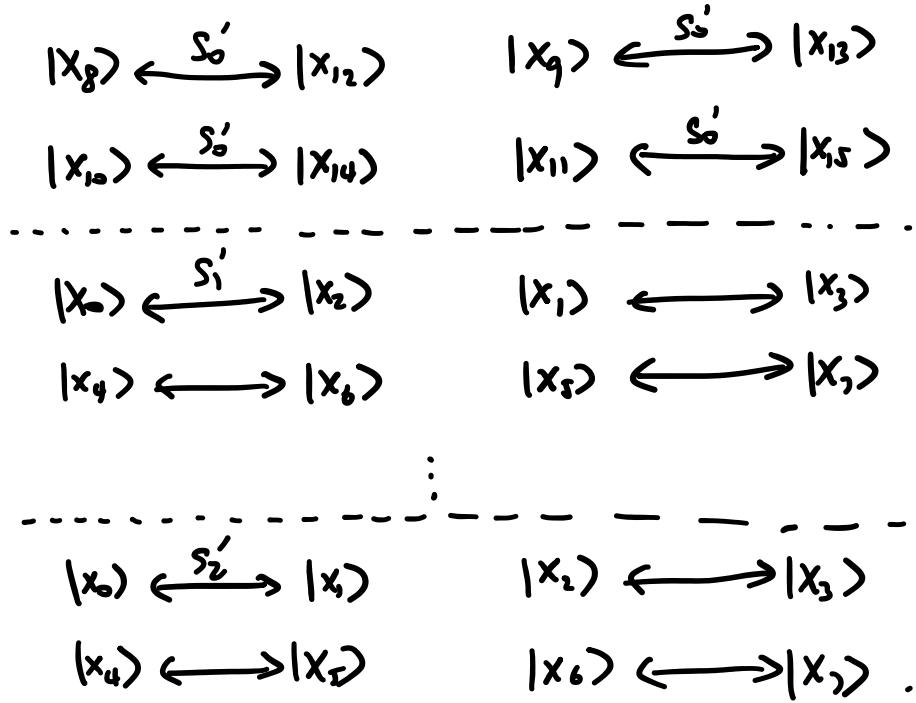
① Effects of S'_0, S'_1, S'_2 on $|x_0\rangle, |x_1\rangle, \dots, |x_5\rangle$.

② Effects of **phase error** on $|x_0\rangle, |x_1\rangle, \dots, |x_5\rangle$.

① Finding **eigenstates** of S'_0, S'_1, S'_2 .

$$|x_0\rangle \xleftrightarrow{S'_0} |x_4\rangle, \quad |x_1\rangle \xleftrightarrow{S'_0} |x_5\rangle$$

$$|x_2\rangle \xleftrightarrow{S'_0} |x_6\rangle, \quad |x_3\rangle \xleftrightarrow{S'_0} |x_7\rangle$$



Eigenstates example:

$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_4\rangle)$: (+)-eigenstates of S_0'

$\frac{1}{\sqrt{2}}(|x_0\rangle - |x_4\rangle)$: (-)-eigenstates of S_0' (because S_0' swaps $|x_0\rangle, |x_4\rangle$)

$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_2\rangle)$: (+)-eigenstates of S_1'

\vdots

② Diagnosing phase errors.

Error example : $E = Z|11111$ "phase flip on qubit 0".

$$|x_0\rangle \xrightarrow{E} |x_0\rangle, |x_1\rangle \xrightarrow{E} -|x_1\rangle$$

$$|x_2\rangle \xrightarrow{E} -|x_2\rangle, |x_3\rangle \xrightarrow{E} |x_3\rangle$$

$$|x_4\rangle \xrightarrow{E} -|x_4\rangle, |x_5\rangle \xrightarrow{E} |x_5\rangle$$

\vdots

$$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_4\rangle) \xrightarrow{\bar{E}} \frac{1}{\sqrt{2}}(|x_0\rangle - |x_4\rangle)$$

S_0' : (+) eigen S_0' : (-) eigen

$\Rightarrow S_0'$ can detect \bar{E} !

$$\text{Similarly, } \frac{1}{\sqrt{8}}(|x_0\rangle + |x_4\rangle + |x_1\rangle + |x_5\rangle + |x_2\rangle + |x_6\rangle + |x_3\rangle + |x_7\rangle)$$

$\downarrow \bar{E}$

$$\frac{1}{\sqrt{8}}(|x_0\rangle - |x_4\rangle + |x_1\rangle - |x_5\rangle + |x_2\rangle - |x_6\rangle + |x_3\rangle - |x_7\rangle)$$

(+/-) eigen of S_0'

$\Rightarrow S_0'$ can still detect \bar{E} !

That seem to be a good candidate for codeword!

$$|0_c\rangle = \frac{1}{\sqrt{8}}(|x_0\rangle + |x_1\rangle + |x_2\rangle + |x_3\rangle + |x_4\rangle + |x_5\rangle + |x_6\rangle + |x_7\rangle)$$

$$|1_c\rangle = \frac{1}{\sqrt{8}}(|x_8\rangle + |x_9\rangle + |x_{10}\rangle + |x_{11}\rangle + |x_{12}\rangle + |x_{13}\rangle + |x_{14}\rangle + |x_{15}\rangle)$$

$|0_c\rangle, |1_c\rangle$ are simultaneous (+1)-eigenstates of S_0, S_1, S_2 and S_0', S_1', S_2' .

Summary : Steane Code ($n=7, k=1$)

Codewords : $|0_c\rangle, |1_c\rangle$

Stabilizers : $S_0, S_1, S_2, S_0', S_1', S_2'$.

It can correct any single-qubit bit-flips and phase-flips!

\Rightarrow any single-qubit errors.

So far, we saw that quantum information can be encoded in a quantum memory, temporarily corrupted by certain errors, and recovered without damage.

Storage of quantum information is safe against noise. ☺

Is processing of quantum information safe?

Fault-Tolerant Computation

(Computation on encoded data) .

Logical X gate. $|0_L\rangle \xrightarrow{X_L} |1_L\rangle$. $|1_L\rangle \xrightarrow{X_L} |0_L\rangle$

Logical Z gate : $|0_L\rangle \xrightarrow{Z} |0_L\rangle$, $|1_L\rangle \xrightarrow{Z} -|1_L\rangle$.

$$x_L = \text{XXXXXXX} \quad , \quad z_L = \text{ZZZZZZZ}$$

Here, x_L , z_L map between codewords.

Sanity check: X_L , Z_L commute with all stabilizers! This is good, because we don't want any stabilizer to flag the logical operations, X_L , Z_L , as errors.

$$\underline{\text{Normalizer}} \quad N(S) = \{ p \in P_n : pg = gp, \forall g \in S \}$$

Here $N(S) = \{11111111, 00000000\} \cup S$.

Remark : ① **Stabilizers** fix every state in codewords

$$|\psi_i\rangle \xrightarrow{g \in S} |\psi_i\rangle \\ = \alpha|0_i\rangle + \beta|1_i\rangle$$

② **Normalizers** map between codewords.

$$|\psi_i\rangle \xrightarrow{h \in N(S)} |\psi'_i\rangle \\ = \alpha'|0_i\rangle + \beta'|1_i\rangle \quad = \alpha'|0_i\rangle + \beta'|1_i\rangle .$$

③ **Errors** (not in S or $N(S)$) take a codeword out of the (+)-eigenstate space.

$$|\psi_i\rangle \xrightarrow{\varepsilon \notin N(S)} |\psi'\rangle \\ = \alpha|0_i\rangle + \beta|1_i\rangle \quad \text{"Something else".}$$

Beyond Pauli Operations? (Next time).