

# Lecture 24 CPSC 447/547 Intro to QC

## Logical Operations

### Outline

- Review Steane Code

- Pauli operations

- Clifford operations.

## Classical Hamming Code ( $n=7$ , $k=4$ )

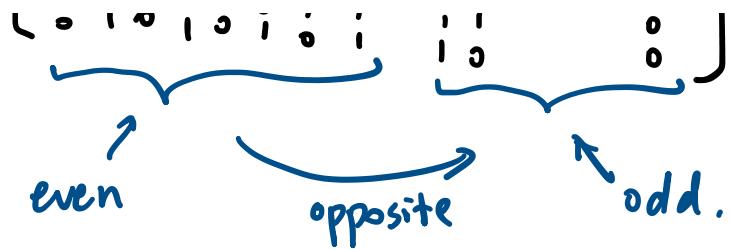
$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Codewords:

$$\begin{bmatrix} 1 \\ x_0 x_1 \dots x_{15} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ \vdots & \vdots \end{bmatrix}$$

$$X = G^T \cdot V.$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & \dots & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \dots & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & \dots & 0 \end{bmatrix}$$



## Steane Code . (n=7, k=1)

Stabilizers :

$$H \rightarrow S = \left[ \begin{array}{c} ZZZZ111 \\ ZZ11ZZ1 \\ Z1Z1Z1Z \\ XXXX111 \\ XX11XX1 \\ X1X1X1X \end{array} \right] \quad \left. \begin{array}{l} s_0 \\ s_1 \\ s_2 \\ s'_0 \\ s'_1 \\ s'_2 \end{array} \right\} \begin{array}{l} Z\text{-type (for correcting bit flips)} \\ X\text{-type (for correcting phase flips)} \end{array}$$

Codewords :

$|X\rangle, |x_1\rangle, \dots, |x_7\rangle$  are (+)-eigenstates of  $s_0, s_1, s_2$ .

Also,  $|\psi\rangle = \sum_j \alpha_j |x_j\rangle$  is ↑

$$S_0 |\psi\rangle = S_1 |\psi\rangle = S_2 |\psi\rangle = |\psi\rangle .$$

Q: What  $|\psi\rangle$  are also (+)-eigenstate of  $s'_0, s'_1, s'_2$  ?

A:

$$\begin{aligned} |\Psi_0\rangle &= |\psi_0\rangle = \sum_{x: \text{IxIeven}} \text{"uniform"} |x\rangle \\ &= \frac{1}{\sqrt{8}} (|x_0\rangle + |x_1\rangle + |x_2\rangle + |x_3\rangle + |x_4\rangle + |x_5\rangle + |x_6\rangle + |x_7\rangle) \end{aligned}$$

$$\begin{aligned}
 |\psi\rangle &= |\psi_L\rangle = \sum_{x: |x| \text{ odd}} \text{"uniform"} |x\rangle \\
 &= \frac{1}{\sqrt{8}} (|x_8\rangle + |x_9\rangle + |x_{10}\rangle + |x_{11}\rangle + |x_{12}\rangle + |x_{13}\rangle + |x_{14}\rangle + |x_{15}\rangle)
 \end{aligned}$$

$|\psi_L\rangle, |\psi_R\rangle$  are simultaneous (+) eigenstates of

$$S_0, S_1, S_2, S'_0, S'_1, S'_2.$$

Diagnosing Errors (for  $|\psi_L\rangle = \alpha|\psi_L\rangle + \beta|\psi_R\rangle$ )

① Single-qubit bit flip.

$$\text{E.g. } E = |X\rangle\langle X|$$

Which stabilizers can detect  $E$ ?

$$|\psi'_L\rangle = E|\psi_L\rangle = \alpha E|\psi_L\rangle + \beta E|\psi_R\rangle$$

$$S_0|\psi'_L\rangle = -|\psi'_L\rangle, S_1|\psi'_L\rangle = -|\psi'_L\rangle, S_2|\psi'_L\rangle = |\psi'_L\rangle$$

★ (-1) eigenstate if  $[S_j, E] \neq 0$  ★

② Single-qubit phase flip

$$\text{E.g. } E = |Z\rangle\langle Z|$$

Which stabilizers can detect  $E$ ?

$$S'_0|\psi'_L\rangle = -|\psi'_L\rangle, S'_1|\psi'_L\rangle = -|\psi'_L\rangle, S'_2|\psi'_L\rangle = |\psi'_L\rangle$$

\* (-1) eigenstate if  $[S_j^z, E] \neq 0$

## Fault-Tolerant Computation

Logical Pauli Operations:

- Logical X gate

$$|0_L\rangle \xrightarrow{X_L} |1_L\rangle, \quad |1_L\rangle \xrightarrow{X_L} |0_L\rangle$$

What's  $X_L$ ?

$$X_L = XXXXXXXX$$

- Logical Z gate.

$$|0_L\rangle \xrightarrow{Z_L} |0_L\rangle, \quad |1_L\rangle \xrightarrow{Z_L} -|1_L\rangle$$

What's  $Z_L$ ?

$$Z_L = ZZZZZZ$$

Sanity check:  $X_L, Z_L$  commute with all stabilizers.

This is important, because we don't want any stabilizers to flag the logical operations as errors.

Normalizer  $N(S) = \{P \in P_n : PG = G P, \forall g \in S\}$

Here  $N(S) = \{XXXXXXX, ZZZZZZZ\} \cup S$

Remark:  $\cap$  **commutes** with normalizers in codimension 1

① **Stabilizers** fix every state in  $\text{span}(S)$ .

$$|\psi_i\rangle \xrightarrow{g \in S} |\psi_i\rangle$$

$$= \alpha |0_i\rangle + \beta |1_i\rangle$$

② **Normalizers** map between codewords

$$\begin{aligned} |\psi_i\rangle &\xrightarrow{h \in N(S)} |\psi'_i\rangle \\ = \alpha |0_i\rangle + \beta |1_i\rangle &= \alpha' |0_i\rangle + \beta' |1_i\rangle \end{aligned}$$

③ **Errors** (not in  $S$  nor  $N(S)$ ) take a codeword out of the codespace.

$$\begin{aligned} |\psi_i\rangle &\xrightarrow{\varepsilon \notin N(S)} |\psi'\rangle \\ = \alpha |0_i\rangle + \beta |1_i\rangle &\quad \text{"something else"} \end{aligned}$$

Is normalizers universal? Unfortunately **NO**: cannot do

### Clifford Operations

Hadamard for example

$$Hx \neq \pm xH$$

$$\begin{aligned} |\psi_i\rangle &\xrightarrow{U \notin P_n} |\psi'_i\rangle ? \\ = \alpha |0_i\rangle + \beta |1_i\rangle &= \alpha' |0'_i\rangle + \beta' |1'_i\rangle \end{aligned}$$

In general,  $|\psi'_i\rangle = U|\psi_i\rangle$  no longer stabilized by  $S$ .

But  $|\psi'_i\rangle$  is stabilized by a new  $S'$ !

E.g.

$$U = H_1$$

$$\xrightarrow{U} |\psi_i\rangle \xrightarrow{\text{action}} |\psi'_i\rangle$$

$$= \alpha|\psi_i\rangle + \beta|l_i\rangle = \frac{\alpha+\beta}{\sqrt{2}}|0_i'\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1_i'\rangle$$

Finding a new set of stabilizers :  $g' \in S'$  for  $U$ .

$$\forall g \in S, U|\psi_i\rangle = Ug|\psi_i\rangle = \underbrace{UgU^+U}_{\text{New stabilizers}}|\psi_i\rangle = g'U|\psi_i\rangle$$

$$\Rightarrow S' = \{UgU^+ \mid g \in S\}$$

It would be great if  $S'$  are still Pauli strings,  $S' \subseteq P_n$ .

Clifford group (on  $n$  qubits) ↳ can use (anti)commutation.

$$C_n = \{U \mid UgU^+ \in P_n, \forall g \in P_n\}$$

$\uparrow$   
n-qubit unitary.

### Example

① Hadamard gate  $H$ .

$$HXH^t = Z, \quad HYH^t = -Y, \quad HZH^t = X$$

② Phase gate  $S = R_z(\frac{\pi}{2})$

$$SXS^t = Y, \quad SYS^t = -X, \quad SZS^t = Z$$

③ CNOT gate

$$CNOT(X \otimes I)CNOT = X \otimes X$$

$$\text{CNOT} (Z \otimes I) \text{CNOT} = Z \otimes I$$

$$\text{CNOT} (I \otimes X) \text{CNOT} = I \otimes X$$

$$\text{CNOT} (I \otimes Z) \text{CNOT} = Z \otimes Z.$$

## Logical Clifford gates on Steane Code.

### ① Logical Hadamard.

$$H_L = H^{\otimes 7} = HHHHHHH$$

↑ Turns out Steane code  
is very nice: all Clifford  
gates can be implemented  
"transversally".

New stabilizers for  $|\psi'_L\rangle = H_L |\psi_L\rangle$

$$S' = \{ H_L g H_L^\dagger \mid \forall g \in S \}$$

"Swapping X-type with Z-type".

New normalizations: What does  $h \in N(S)$  do to  $|\psi'_L\rangle$ ?

$$\begin{cases} X_L \longrightarrow H_L X_L H_L = Z_L \\ Z_L \longrightarrow H_L Z_L H_L = X_L \end{cases}$$

Acting  $X^{\otimes 7}$  on  $|\psi'_L\rangle$  is the new logical Z gate.

$$\dots Z^{\otimes 7} \quad \dots \quad X \quad \dots$$

### ② Logical Phase gate

$$\text{Is it } S_L = S^{\otimes 7}?$$

$$S^{\otimes 2} X^{\otimes 2} S^{*\otimes 2} = Y^{\otimes 2} = i^2 X^{\otimes 2} Z^{\otimes 2} = -i X_L Z_L = -Y_L$$

$$\text{So } S_L^+ = S^{\otimes 2}.$$

### ③ Logical CNOT gate

$$X_L \otimes I = X^{\otimes 2} \otimes I^{\otimes 2} \xrightarrow{\text{CNOT}^{\otimes 2}} X^{\otimes 2} \otimes X^{\otimes 2} = X_L \otimes X_L$$

$$I_L \otimes Z_L = I^{\otimes 2} \otimes Z^{\otimes 2} \xrightarrow{\text{CNOT}^{\otimes 2}} Z^{\otimes 2} \otimes Z^{\otimes 2} = Z_L \otimes Z_L$$

...

Turns out H, S, CNOT are generators of  $C_n$ .

But still not universal! Need one additional gate for universality.